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The dynamical influence of large-scale heat sources and sinks on the quasi-stationary mean motions of the atmosphere

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SUMMARY

The classical convective theory for the influence of large-scale zonally asymmetric heating and cooling on the quasi-stationary mean motions is re-examined and shown to be inconsistent with observation. Scale considerations indicate that the classical mechanism cannot be applied to heating on a scale larger than that responsible for sea-breezes. In analogy with the theory of orographically produced stationary disturbances, heat sources and sinks of planetary dimensions produce quasi-static and quasi-geostrophic perturbations. A thermally active model is constructed in which all motions except those of large scale are filtered out *a priori*. The small perturbations produced by an idealized large-scale distribution of heat sources and sinks acting on a horizontally uniform baroclinic zonal current confined between rigid horizontal planes are then studied. The influence of surface friction is taken into account. Calculations are also performed on a model with no friction and on the assumption that entropy is advected horizontally. The results generally show good agreement with those for the more general model. When the stratosphere is taken into account, the disturbances in the upper troposphere are reduced in magnitude. The conditions for resonance in the absence of friction are determined. The passage through the resonant state is shown to modify significantly the character of the motions. The observed normals are discussed at some length in light of the theoretical computations. It is found that the magnitude and spatial distribution of the model disturbances display substantial correspondence with observation when allowance is made for orographic effects. Finally a method is suggested for quantitatively deducing the three-dimensional field of non-adiabatic heating and cooling from the observed normal maps by dynamical considerations.

1. INTRODUCTION

In an attempt to account for weather changes over periods of the order of 10 to 30 days, it seems logical to try first to understand the accompanying evolutions in the large-scale flow pattern. One of the more successful techniques for long-range forecasting has involved subjective predictions of the variations of tropospheric flow patterns averaged in time over a period of 30 days (Namias 1948). The disturbances on 30-day mean maps are observed to change in a slow and regular manner. Furthermore, it has been observed by Namias (1952) that from December to March and from June to September the departures from normal of these monthly mean maps exhibit a great deal of persistence, with rapid changes occurring just after the equinoxes. This persistence for periods of about four months in midwinter and midsummer is also noted in normal flow patterns. It therefore appears that an explanation of the normal motions would be of fundamental value in deducing the mechanism of the transient motions.

Examination of daily upper-air maps shows that those components of the long planetary waves containing most of the perturbation energy travel with a phase speed so large that one would expect the averaging process to produce a substantial reduction in their amplitudes. The fact that disturbances of considerable amplitude still appear on the averaged maps would seem to imply that the stationary components of the wave motion are excited by geographically fixed sources of disturbance. Variations from month to month in the perturbation pattern of the monthly mean maps could be attributed to changes of the position and intensity of the sources of disturbance, or to changes in the zonally averaged properties of the flow, or to a combination of both. It has been suggested

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by various investigators that orography (e.g., Rossby 1939, Haurwitz 1940, Charney and Eliassen 1949, Bolin 1950), heat sources and sinks (e.g., Rossby 1939, Haurwitz 1940, Sutcliffe 1951, Bleeker 1950, Wippermann 1951), or variable surface friction at coast lines (e.g., Eady and Sawyer 1951) are responsible for at least a portion of these disturbances.

The role of mountains in determining the normal motions has been studied quantitatively by Charney and Eliassen (1949). They considered the orographically induced steady motions along a single latitude of an equivalent barotropic atmosphere of finite lateral extent and with a constant zonal velocity. The basic current was assumed to be unaffected by friction but the perturbations were subjected to a spatially constant eddy diffusivity at the lower boundary of the atmosphere. This model, when applied to the January normal 500-mb flow at 45° latitude, displayed a surprising degree of success in predicting the magnitude and position of the observed disturbances despite the rather restrictive kinematical and dynamical constraints placed upon the motions. Charney and Eliassen discounted the influence of heat sources and sinks on the 500-mb flow on the grounds that no reversal in phase of the fixed perturbations is observed to occur at this level from winter to summer.

Since Charney and Eliassen found the effects of friction to be of considerable importance, there remained the possibility that the spatial variation of the earth's roughness might produce important modifications in the motion. The present writer (in an unpublished study) has investigated the effect of variable surface friction on the orographically produced disturbances at 500 mb and has found that it results in a change of only about 10 per cent from the case of constant friction. This is to be expected since friction only influences the motions appreciably if it is acting over a long period of the air parcel's history of travel. Charney and Eliassen found that if the constant eddy diffusivity were increased fourfold everywhere the change in the stationary flow would be considerable. However, if the eddy coefficient increases by this much or even more in passing over the west coast of a continent of 5,000 km to 10,000 km breadth, but decreases again at the east coast, then in the equilibrium state the resulting motions will not differ by much from those for which the underlying surface was uniformly rough with a coefficient of friction equal to the average value in the variable case.

There is still considerable speculation about the manner in which the energy of solar radiation is made available for driving the atmosphere. That the atmosphere is relatively transparent to the incoming short waves means that the major portion of this energy is not *directly* utilized. Hence there will be a meridional gradient of heating due not only to the variation of angle of incidence, but also to the meridional variation of the zonally-averaged properties of the atmosphere and earth's surface below. The zonal mean potential energy thus acquired is then transformed to maintain the general circulation, probably by some mechanism which depends on the hydrodynamic stability properties of the flow itself. However, it is the purpose of the present study to determine how zonal asymmetries of heating affect the quasi-stationary perturbations. The use of linear arguments will make it possible to consider this problem separately, in the first approximation, from that of the maintenance of the general circulation.

Some investigators have assumed that differential heating generates meridional cellular circulations of the Hadley type which serve to maintain the zonal kinetic energy of the general circulation. However, it has been suggested by Rossby (1949) and Starr (1951) on the basis of recent observational evidence that the maintenance of the general circulation is accomplished primarily by horizontal exchange processes. Thermal circulations in vertical planes, classically, have also been postulated as the mechanism by which zonally asymmetric heat sources and sinks in middle latitudes produce the large-scale troughs and ridges which appear on normal maps, especially the monsoonal flow patterns. The classical theory may be traced back as far as 1686, when Halley accounted for the trade winds and monsoons by the ascent of warm air in the hotter regions of the globe. Hadley, in 1735, modified Halley's explanation by taking into account the influence of the earth's rotation. The classical reasoning parallels the theory which is used to account for land and sea breezes (see for instance Willett 1944).

In the case of the Asiatic winter monsoons, it follows from the classical argument that the surface Siberian anticyclone will occur at the place of maximum cooling. From the first law of thermodynamics it can be seen that at the surface, where the vertical velocity is zero, the heat change per unit mass in the steady state is proportional to the horizontal temperature advection. One can thus deduce the locations of those heat sources or sinks in the lowest kilometre. Inspection of the January normal sea-level map (Fig. 6 (a)) indicates that the maximum warm-air geostrophic advection (which should balance the external non-adiabatic cooling) is not found at the maximum pressure but approximately 30° to the west. This phase lag may also be observed in the case of the heating off the east coasts of Asia and North America in winter and the Aleutian and Icelandic depressions, respectively. The maximum temperature, however, is found to the east of the low centres and the minimum temperature is found to the east of the high centres. The classical mechanism therefore does not seem to explain what is observed. It is proposed to show in this study that the apparent success of the usual monsoon explanation is due to a fortunate compensation of two errors, namely, a faulty primary mechanism applied to an incorrectly placed field of heat sources and sinks.

2. The effect of the scale of heating and cooling on the dynamics of the motions

Since the atmosphere is capable of sustaining a wide spectrum of disturbance frequencies, it seems reasonable to ask whether the forces which primarily govern the propagation of the disturbances are the same for all scales. That this is not so has been realized intuitively by meteorologists for many years. Charney (1948), in a recent theoretical investigation, has not only demonstrated which forces are irrelevant as far as disturbances of planetary proportions are concerned but also offered a rationale for the *a priori* filtering of the equations of motion so that they would apply only to motions of planetary dimensions. For our present purpose we would like to know, for an external source of disturbance of certain scale, what forces will be primarily responsible for the nature of the perturbation which is created in the flow ? A rather complete analysis has been made by Queney (1948) for the case of orographically produced disturbances. He has shown that in a stratified rotating atmosphere moving with a constant zonal speed U, a number of distinct types of stationary waves can be produced by a cylindrical mountain range whose length along a meridian is much greater than its half-width δ :

(i) If $\delta \simeq 1$ km to 10 km, then short stationary gravity waves are produced whose wavelength is determined by U and the static stability. The rotation of the earth is unimportant. The disturbance does not contain a meridional component of motion.

(ii) If $\delta \simeq 100$ km, a complex system of stationary gravity-inertia waves is initiated with a horizontal wavelength depending on the Coriolis parameter and U, but with a vertical wavelength determined by U and the static stability. These waves already behave quasi-statically (Queney 1947, p. 63). There is now a meridional component of motion, but it is not geostrophic. The flow is quasi-horizontal.

(iii) If $\delta \simeq 1{,}000$ km, there is a system of quasi-static and quasi-geostrophic planetary waves with length essentially determined by U and the latitudinal variation of the Coriolis parameter. The flow is quasi-horizontal.

The above results should apply equally well to other types of sources, and in particular to heat and cold sources. For instance, Malkus and Stern (1951) and Stern and Malkus (1952) have studied the motions resulting from an infinite-strip island 10 km in width and normal to the mean flow. The resulting mathematical model is essentially the same as those constructed by Queney (1947) and Scorer (1949) for the mountain-wave problem of scale comparable to that given in (i). Therefore, on the basis of Queney's classification, one can infer a corresponding classification of heat sources and sinks with respect to the types of stationary disturbances that they will generate :

(i) Heating on the scale that produces convection and small-scale sea breezes.

(ii) Continental coastal heating which induces large-scale sea breezes of the type considered by Haurwitz (1947).

(iii) Heat sources and sinks of planetary dimensions which produce large-scale disturbances such as the monsoons. These will be dealt with in the present paper.

3. The observed distribution of heat sources and sinks

The atmosphere receives or gives off heat essentially in the following ways :

(i) Release of latent heat through condensation within the atmosphere. This heat is made available by the transport of water vapour through the lower boundary.

(ii) Turbulent transport of sensible heat through the lower boundary to or from the earth's surface.

(iii) Long-wave radiation through the lower boundary to or from the earth's surface.

(iv) Short-wave radiation through the upper boundary from the sun.

(v) Long-wave radiation through the upper boundary to space.

Horizontal turbulent transport of heat *within* the atmosphere is not included since it serves only to redistribute heat energy which has already been absorbed by the atmosphere.

Since the various types of heating cannot be observed directly, they must be deduced by means of theoretical relationships from physical quantities which are directly measurable. Incomplete theory and inadequate observation have hampered efforts to determine in detail the seasonal, geographical, and vertical distributions of the sources and sinks of heat in the atmosphere. However, some progress has been made and much research at present is being directed toward this problem.

The release of latent heat of condensation in a vertical air column is perhaps the simplest of the sources to evaluate. Although heat is released during cloud formation, many clouds ultimately evaporate, thus absorbing an equivalent amount of heat from the atmosphere. Since these clouds move with the air motion there is no net heating. We assume that precipitation occurs soon after condensation. Therefore, a measure of the heat permanently given up to the atmosphere is the precipitation, P, reaching the ground. The amount of heat is $S_p = L_t P$, where L_t is the latent heat of condensation. If the precipitation reaches the ground as snow, then the latent heat of sublimation must be used instead of L_t. Jacobs (1951) has computed S_p seasonally for the oceans of the northern hemisphere. This may be done similarly for the continents from the rainfall data given by Landsberg (1945). A composite chart of the heat of condensation liberated for the winter season December to February is shown in Fig. 1 (a) and for the summer season June to August in Fig. 1 (b). Since the assumption that all precipitation is in the form of rain results in an error of only about 12 per cent when snow falls (Jacobs 1949), this difference was ignored.

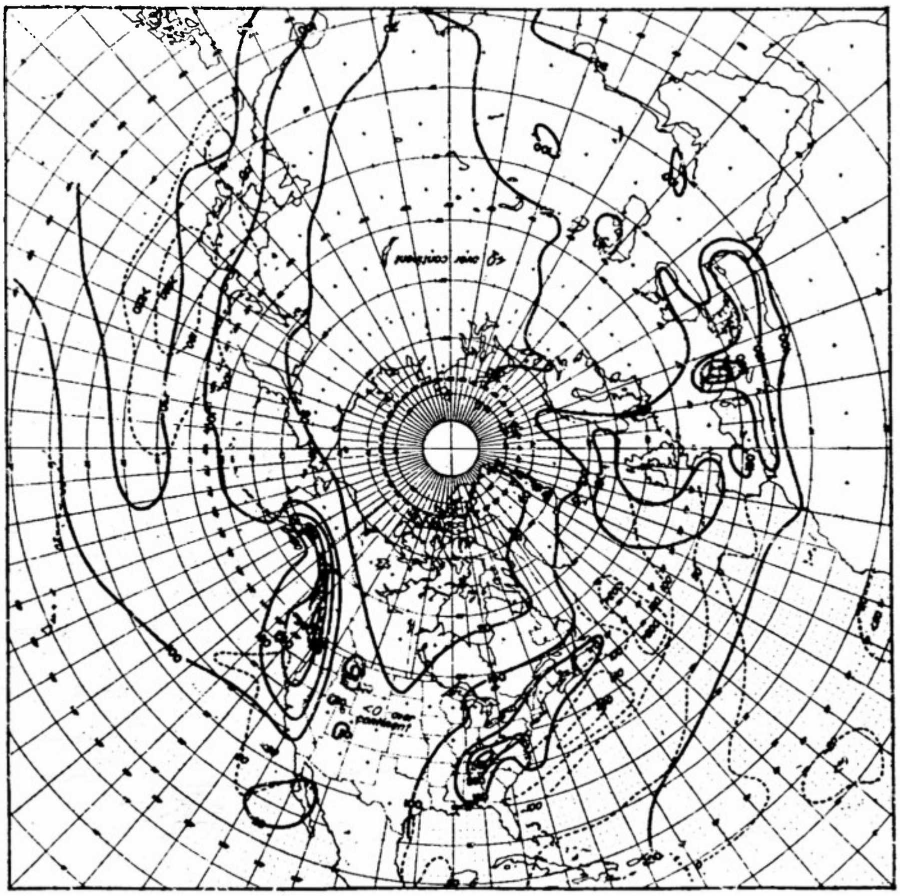

Figure 1 (a). Winter : December to February.

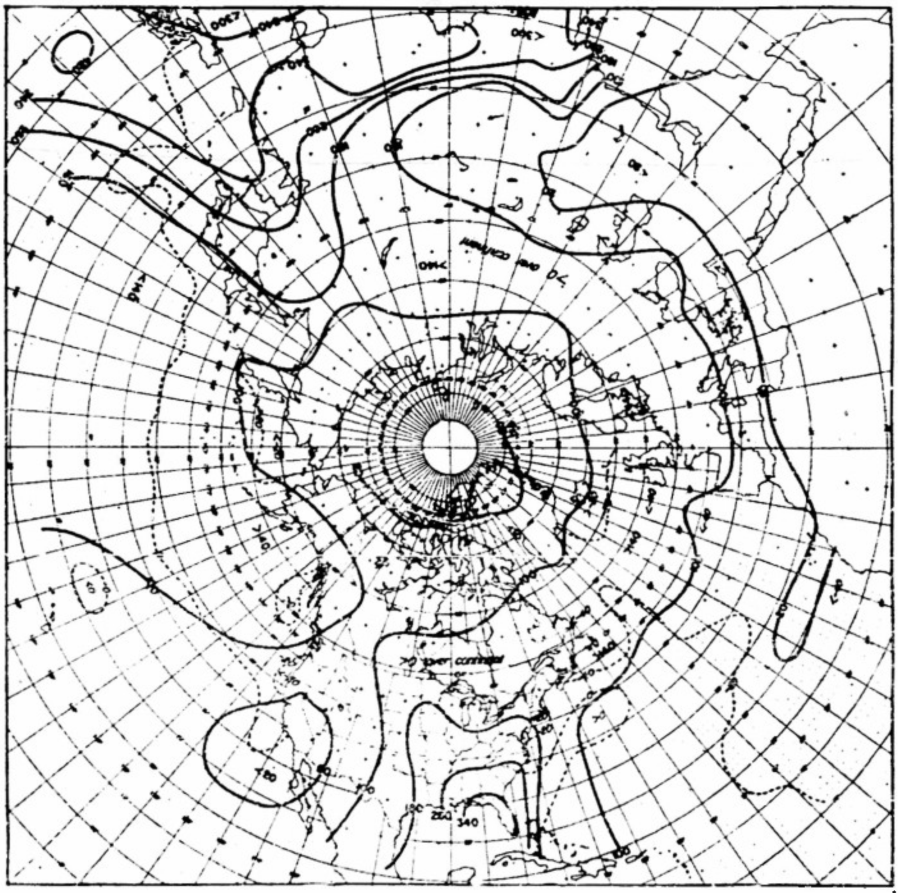

Figure 1 (b). Summer : June to August.

Figure 1. Normal seasonal non-adiabatic heating and cooling (cal cm^{-2} day^{-1}). Heat of condensation is shown by solid lines and bold type, vertical eddy conduction of sensible heat by dashed lines and italics. Oceanic data after Jacobs (1951), continental data are deduced from Landsberg (1945).

Jacobs (1943) has also calculated the sensible heat transported into the atmosphere from the sea surface; this is also shown in Figs. 1 (a) and 1 (b). Variations in heat thus made available over the oceans are primarily due to the warm and cold ocean currents. Calculations of seasonal eddy conduction over land are not available, although numerous local studies are being conducted, particularly in England and Australia. Generally, one may say that the eddy flux of heat over continental masses is downward in winter and upward in summer.

It has been shown (Elsasser 1942 and Panofsky 1947) that heat changes due to nocturnal radiation are small compared with the turbulent transfer of heat and are negligible at heights above about 50 m. The exception may occur over large continental masses which are snow-covered, for instance, Asia and Canada in the winter. Here the net radiative flux is from the air in the lowest layers to the ground (Wexler 1936) and thus is in the same direction as the eddy flux of sensible heat in winter.

The short-wave radiation from the sun is absorbed by ozone, water vapour, clouds, and by solid impurities in the atmosphere. Although the amount of this heating is appreciable, it is probable that its variation with longitude is small compared with the variations due to condensation, eddy conduction, and long-wave radiation from the earth's surface. It will be shown that in the model considered only those sources and sinks which have appreciable deviations from their zonal means give rise to disturbances in the zonal flow.

The net upwardly directed infra-red flux is considerable and is a major heat sink of the atmosphere for all seaons at all latitudes. However, this agency is also relatively uniform along latitudes.

Möller (1950a) has calculated the vertical distribution of heat added to or removed from the atmosphere in middle latitudes due to condensation (corresponding to an annual precipitation of 1,000 mm) and to radiative heat transfer under the condition of mean cloudiness. Möller's calculations show relatively little heating at the ground due to condensation; however, the amount of heating increases rapidly upward to a maximum at about 2 km and then decreases again to zero at 10 km. We also know that the magnitude of heating or cooling due to vertical eddy conduction and nocturnal radiation at any one place is a maximum at the bottom of the atmosphere and decreases rapidly with height. It appears, therefore, that the sources and sinks of heat which are mainly responsible for producing disturbances in the flow, occur at or below the condensation level, i.e., roughly 6 km.

Thus far we have attempted to describe, on the basis of existing direct or indirect observational evidence, where the various agencies inject or remove heat from the atmosphere. In motions that are averaged in time, Reynolds eddy stresses are generated which serve to redistribute momentum and heat. The magnitude of these stresses is determined by the degree of correlation of the velocity components with each other and with entropy. The modification produced by large-scale mixing on the influence of the heat sources and sinks on the normal motions has been considered by Elliott and Smith (1949) and Möller (1950b). In the present study only the direct three-dimensional heating effects, in idealized form, are calculated. The variations with longitude of the large-scale eddy stresses will not be considered.

Of course, the heat sources and sinks described above are determined to some extent by the disturbed motion itself. Certainly the distribution of water vapour, cloudiness, precipitation, static stability, and the field of motion interact to produce the average conditions. Since we are dealing with stationary motions it is permissible to specify the field of heating and cooling, and the problem will be to find the field of motion which is dynamically consistent with it.

4. A THEORETICAL HEATING MODEL

A stationary zonally-symmetric flow subjected to steady zonally-symmetric heating and cooling will eventually reach an equilibrium state that is also zonally-symmetric. On the other hand, if the sources and sinks of heat vary longitudinally, then we must expect disturbances on the zonal current. The observational evidence in Section 3 indicates that some of the sources and sinks of heat have appreciable zonal asymmetry, especially those that exist in the lowest 6 km of the atmosphere. We shall idealize the horizontal distribution of perturbation heating by assuming a periodic variation in the latitudinal and longitudinal directions.

With the assumption that $w\, \partial u/\partial z$ and $w\, \partial v/\partial z$ in the horizontal equations of motion are small compared with the horizontal advection (Charney 1948), the vorticity equation reads

$$\frac{d_h(\zeta + f)}{dt} + (\zeta + f)\, \nabla_h \cdot \mathbf{v}_h = -\mathbf{k} \cdot \nabla_h \left(\frac{1}{\rho}\right) \times \nabla_h\, p, \qquad (1)$$

where d_h/dt is the individual horizontal time-rate of change following a material particle, ∇_h is the horizontal gradient, $\mathbf{v}_h$ is the horizontal velocity vector, ζ is the vertical component of relative vorticity, f is the Coriolis parameter, $\mathbf{k}$ is the vertical unit vector, ρ is the density, p is the pressure. With use of the geostrophic approximation, the right side, which represents the pressure volume solenoids with vertical axes, becomes $-f\,\mathbf{v}_h \cdot \nabla_h \ln \rho$. We assume small perturbations on a zonal current U which is constant with latitude and varies linearly with height, i.e., $U(z) = U(0) + \Lambda z$. Upon applying the equation of continuity and requiring steady motions, we have from Eq. (1) :

$$U \frac{\partial \zeta}{\partial x} + \beta v = \frac{f}{\rho} \frac{\partial (\rho w)}{\partial z}.$$

Here v, w, and ζ are perturbation quantities, ρ and U are zonally averaged quantities, and $\beta = df/dy$. x is positive eastward, y is positive northward, z is positive upward and u, v, and w are the respective velocity components. Hereafter the variation of f is insignificant and f will therefore be assumed constant. Upon evaluating $\partial\zeta/\partial x$ geostrophically we have

$$U\, \nabla_h^2 v + \beta v = \frac{f}{\rho} \frac{\partial (\rho w)}{\partial z} \qquad (2)$$

We can obtain an additional differential relation between v and w which involves the non-adiabatic heating. From the first law of thermodynamics and the definition of potential temperature θ we have

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_h \cdot \nabla_h + w \frac{\partial}{\partial z}\right) \ln \theta = \frac{1}{c_p T} \frac{dq}{dt} \equiv Q \qquad (3)$$

in which T is the temperature, dq/dt is the rate of change of heat per unit mass, c_p is the specific heat of air at constant pressure and thus $c_p\, Q$ is the rate of change of entropy per unit mass. Eq. (3) will be referred to as the *thermodynamic energy equation*. The thermal wind may be approximated in terms of θ :

$$\frac{\partial \mathbf{v}_h}{\partial z} \simeq \frac{g}{f} \mathbf{k} \times \nabla_h \ln \theta \qquad (4)$$

For steady motions (3) and (4) give

$$\left(u \frac{\partial v}{\partial z} - v \frac{\partial u}{\partial z}\right) + \frac{g}{f} \frac{\partial \ln \theta}{\partial z} w = \frac{gQ}{f} \qquad (5)$$

Again assuming small perturbations, Eq. (5) becomes

$$U\frac{\partial v}{\partial z} - \Lambda v + \frac{g}{f}\frac{\partial \ln \theta}{\partial z} w = \frac{gQ}{f} \qquad (6)$$

Here Q is a perturbation quantity, and $\partial \ln \theta/\partial z$ is a mean quantity.

We assume a constant lapse rate γ,

$$T(y, z) = T(y, 0) - \gamma z \qquad (7)$$

However, the vertical variation of the zonally-averaged pressure is given to sufficient accuracy in terms of the mean temperature $\overline{T}$ by

$$p(y, z) \simeq p(y, 0) \exp(-z/H), \; H \equiv R\overline{T}/g \qquad (8)$$

The vertical variation of the zonally-averaged density may be determined by differentiating the equation of state with respect to z, then using (7) and (8) and integrating again with the assumption that $\overline{T}$ may replace T when it appears as a coefficient.

$$\rho(y, z) \simeq \rho(y, 0) \exp(-z/H^*), \; H^* \equiv \overline{T}\Big/\left(\frac{g}{R} - \gamma\right) \qquad (9)$$

where R is the gas constant for air. Hence the frequency of a buoyancy oscillation ν is given by,

$$\nu^2 \equiv g\frac{\partial \ln \theta}{\partial z} \simeq \frac{g}{\overline{T}}(\gamma_d - \gamma), \qquad (10)$$

where γ_d is the adiabatic lapse rate.

We now assume the perturbation heat sources and sinks to vary sinusoidally with x and y:

$$Q(x, y, z) = \chi(z) \sin kx \sin \mu y \qquad (11)$$

where $k = 2\pi/L$ and $\mu = 2\pi/D$, L and D being respectively the longitudinal and the lateral wavelengths. Consistent with this variation, v and w must have the form

$$\left.\begin{aligned} v(x, y, z) &= [V_1(z) \sin kx + V_2(z) \cos kx] \sin \mu y \\ w(x, y, z) &= [W_1(z) \sin kx + W_2(z) \cos kx] \sin \mu y \end{aligned}\right\} \qquad (12)$$

in which $V_1(z)$, $V_2(z)$, $W_1(z)$, and $W_2(z)$ are to be determined. Upon substituting (11) and (12) into (2) and (6) and equating coefficients of $\sin kx$ and $\cos kx$, there results a system of two pairs of ordinary differential equations which can be solved simultaneously to eliminate $W_1(z)$ and $W_2(z)$. We obtain

$$\left.\begin{aligned} \frac{d^2V_1}{dz^2} - \frac{1}{H^*}\frac{dV_1}{dz} - a^2\left(\frac{U - U_c^*}{U}\right) V_1 &= \frac{g}{f\rho U}\frac{d(\rho\chi)}{dz} \\ \frac{d^2V_2}{dz^2} - \frac{1}{H^*}\frac{dV_2}{dz} - a^2\left(\frac{U - U_c^*}{U}\right) V_2 &= 0 \end{aligned}\right\} \qquad (13)$$

where we have abbreviated:

$$\left.\begin{aligned} U_c &= \beta/(\mu^2 + k^2) \\ a^2 &= (\mu^2 + k^2)\nu^2/f^2 \\ U_c^* &= U_c + \Lambda/(a^2H^*) \end{aligned}\right\} \qquad (14)$$

We shall consider the effect of surface friction, but only a[illegible] acts on the perturbations. The energy dissipated in this manner is a second-order quantity. Since we have taken a periodic distribution of relative heat sources and sinks, there is no net heat-energy contribution. Therefore, the frictional loss of energy must be balanced by appropriate second-order changes in the energy of the basic current. Thus there is no ambiguity in assuming that the zonal velocity is steady to the first order of small quantities.

The introduction of surface friction into the baroclinic model may be accomplished by assuming the friction layer to be a barotropic boundary layer. The justification of this lies in the fact that the variation of the wind with height due to baroclinicity in the friction layer is probably small compared with that due to the surface stress. As has been demonstrated by Charney and Eliassen (1949), vertical velocities are generated at the top of a barotropic friction layer which are proportional to the perturbation vorticity at this level. If we now let $z = 0$ be this height, then the above statement constitutes a lower boundary condition for our model :

$$w(x, y, 0) = HF\zeta(x, y, 0)/f \quad . \quad . \quad . \quad . \quad . \quad (15)$$

in which

$$F = \frac{\sin 2\alpha}{H}\left(\frac{Kf}{2}\right)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

α is the angle between the isobars and the surface wind, and K is the eddy diffusivity. F will be taken as constant and we will adopt the mean values suggested by Brunt (1939) : $\alpha = 22{\cdot}5°$ and $K = 10\ \mathrm{m^2\,sec^{-1}}$, which with $f = 10^{-4}\ \mathrm{sec^{-1}}$ and the height of the homogeneous atmosphere $H = 8$ km yields $F = 2{\cdot}0 \times 10^{-6}\ \mathrm{sec^{-1}}$.

For the upper boundary condition it would be desirable to simulate the stabilizing influence of the stratosphere on the troposphere without taking the stratosphere into account explicitly. A rigid top, which is equivalent to a stratosphere of infinite static stability, would result in large mass divergences just below the tropopause. This is especially true if an extremum in vertical velocity occurs near the upper boundary. We shall first assume a rigid top : $w = 0$ at $z = z_T$. We therefore must expect that the amplitudes of the disturbance in the upper portion of the model will be somewhat larger than if we had assumed a moderately stable fluid layer in place of the rigid top.

Using (11) and (12) in the energy equation (6) and imposing the boundary conditions (15) and $w = 0$ at $z = z_T$, we have at $z = 0$:

$$\left.\begin{aligned} U\frac{dV_1}{dz} - \Lambda V_1 - \frac{\nu^2 HF(\mu^2 + k^2)}{f^2 k} V_2 &= \frac{g\chi}{f} \\ U\frac{dV_2}{dz} - \Lambda V_2 + \frac{\nu^2 HF(\mu^2 + k^2)}{f^2 k} V_1 &= 0 \end{aligned}\right\} \quad . \quad (17)$$

and at $z = z_T$:

$$\left.\begin{aligned} U\frac{dV_1}{dz} - \Lambda V_1 &= \frac{g\chi}{f} \\ U\frac{dV_2}{dz} - \Lambda V_2 &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Let $a^{*2} = a^2 + (2H^*)^{-2}$, then system (13), (17) and (18) may be transformed by

$$\left.\begin{aligned} \phi_1(z) &= V_1(z)\exp\left[(a^* - (2H^*)^{-1})\,z\right] \\ \phi_2(z) &= V_2(z)\exp\left[(a^* - (2H^*)^{-1})\,z\right] \\ \xi\ (z) &= 2a^*(z + \Lambda^{-1}U(0)) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (19)$$

to give

$$\left.\begin{aligned} \xi\phi_1'' - \xi\phi_1' + r\phi_1 &= \Pi(\xi) \\ \xi\phi_2'' - \xi\phi_2' + r\phi_2 &= 0 \end{aligned}\right\}, \quad \xi_0 \leqslant \xi \leqslant \xi_T \qquad (20)$$

$$\left.\begin{aligned} \phi_1' - \lambda\phi_1 - \sigma\phi_2 &= \kappa \\ \phi_2' - \lambda\phi_2 + \sigma\phi_1 &= 0 \end{aligned}\right\}, \quad \xi = \xi_0 \qquad (21)$$

$$\left.\begin{aligned} \phi_1' - \lambda\phi_1 &= \kappa \\ \phi_2' - \lambda\phi_2 &= 0 \end{aligned}\right\}, \quad \xi = \xi_T \qquad (22)$$

where a prime indicates ordinary differentiation with respect to ξ and we have abbreviated:

$$\left.\begin{aligned} r &= a^2\, U_c^*/(2\, a^*\, \Lambda) \\ \Pi(\xi) &= \frac{g\,(\rho\chi)'}{f\Lambda\,\rho} \exp\left[\left(\tfrac{1}{2} - \frac{1}{4a^*\,H^*}\right)\left(\xi - \xi_0\right)\right] \\ \lambda(\xi) &= \tfrac{1}{2} - \frac{1}{4a^*\,H^*} + \frac{1}{\xi} \\ \kappa(\xi) &= g\chi(\xi)/(f\,\Lambda\,\xi) \\ \sigma &= \nu^2\, HF\,(\mu^2 + k^2)/(kf^2\,\Lambda\,\xi_0) \end{aligned}\right\} \qquad (23)$$

The homogeneous portion of (20) is of the same form as the differential equation derived by Charney (1947) in connection with a study of baroclinic instability. In this paper he gave a thorough discussion of the solutions to this form of the confluent hypergeometric differential equation. With Charney's notation $\psi_2(\xi, r)$ is the solution of the *first* kind and $\psi_1(\xi, r)$ is a linearly independent solution of the *second* kind.

The general solutions of (20) are

$$\left.\begin{aligned} \phi_1(\xi) &= A(\xi)\,\psi_1(\xi) + B(\xi)\,\psi_2(\xi) \\ \phi_2(\xi) &= E_1\,\psi_1(\xi) + E_2\,\psi_2(\xi) \end{aligned}\right\} \qquad (24)$$

in which

$$\left.\begin{aligned} A(\xi) &= \int_{\xi_0}^{\xi} \frac{\Pi(\xi)\,d\xi}{\xi\,\psi_2\,(\psi_1/\psi_2)'} + A(\xi_0) \\ B(\xi) &= -\int_{\xi_0}^{\xi} \frac{\psi_1\,\Pi(\xi)\,d\xi}{\xi\,\psi_2^2\,(\psi_1/\psi_2)'} + B(\xi_0) \end{aligned}\right\} \qquad (25)$$

$A(\xi_0)$, $B(\xi_0)$, E_1, and E_2 are fixed by the boundary conditions and may be determined by applying (24) and (25) to (21) and (22):

$$\left.\begin{aligned} E_1 &= \frac{C_4\left[(C_4\,C_7 - C_3\,C_8)\,(C_3\,\kappa(\xi_0) + C_1\,G) + (C_2\,C_3 - C_1\,C_4)\,C_7\,G\right]}{C_3\left[(C_2\,C_3 - C_1\,C_4)^2 + (C_4\,C_7 - C_3\,C_8)^2\right]} \\ E_2 &= -\frac{C_3\,E_1}{C_4} \\ B(\xi_0) &= \frac{C_3\,(C_1\,E_1 + C_2\,E_2) + C_7\,G}{C_4\,C_7 - C_3\,C_8} \\ A(\xi_0) &= \frac{G - C_4\,B(\xi_0)}{C_3} \end{aligned}\right\} \qquad (26)$$

where

$$\left.\begin{aligned} C_1 &= \psi_1'(\xi_0) - \lambda(\xi_0)\,\psi_1(\xi_0) \\ C_2 &= \psi_2'(\xi_0) - \lambda(\xi_0)\,\psi_2(\xi_0) \\ C_3 &= \psi_1'(\xi_T) - \lambda(\xi_T)\,\psi_1(\xi_T) \\ C_4 &= \psi_2'(\xi_T) - \lambda(\xi_T)\,\psi_2(\xi_T) \\ C_5 &= \quad A(\xi_T) - A(\xi_0) \\ C_6 &= -\left[B(\xi_T) - B(\xi_0)\right] \\ C_7 &= \sigma\,\psi_1(\xi_0) \\ C_8 &= \sigma\,\psi_2(\xi_0) \\ G &= \kappa(\xi_T) - C_3 C_5 + C_4 C_6 \end{aligned}\right\} \quad . \quad . \quad . \quad (27)$$

In order to simulate the observed vertical variation of heating we choose an analytic expression for χ satisfying the requirement that it have a maximum in mid-troposphere and vanish at the tropopause :

$$\chi = N\,e^{-z/h} \sin(\pi z / z_T) \quad . \quad . \quad . \quad . \quad (28)$$

In particular, this form of χ vanishes at $z = 0$ and z_T, and thus $\kappa(\xi_0) = \kappa(\xi_T) = 0$. The altitude of maximum χ is controlled by h. Our knowledge of the details of χ is rather scanty; however, we do have a good estimate of the perturbation total heat change S in a column of unit area. This quantity is related to Q by :

$$S = \int_0^{z_T} \rho \frac{dq}{dt}\,dz = \int_0^{z_T} c_p\,T\rho Q\,dz \simeq c_p\,\overline{T} \int_0^{z_T} \rho Q\,dz \quad . \quad . \quad (29)$$

The maximum value of S in the horizontal, using (11), is thus

$$S_{\max} = c_p\,\overline{T} \int_0^{z_T} \rho\chi\,dz \quad . \quad . \quad . \quad . \quad (30)$$

If we wish to vary h in (28) but keep $S_{\max}$ invariant, N must be determined by

$$N = \frac{S_{\max}}{c_p\,\overline{T}\,\rho(0)} \frac{\left(\dfrac{1}{h} + \dfrac{1}{H^*}\right)^2 + \left(\dfrac{\pi}{z_T}\right)^2}{\left(\dfrac{\pi}{z_T}\right)\left\{1 + \exp\left[-\left(\dfrac{1}{h} + \dfrac{1}{H^*}\right) z_T\right]\right\}} \quad . \quad . \quad (31)$$

Figs. 1 (a) and 1 (b) indicate that the vertical integral of the heat change is of the order of magnitude of 100 cal cm^{-2} day^{-1}. The elements of greatest uncertainty are the magnitude of the sensible heat transfer over the continents and the magnitude of the nocturnal radiation in winter over Asia and Canada. What little is known about these factors (see, for instance. London 1952) indicates that they are at most of the same order of magnitude as the contributions from heat of condensation. The value of $S_{\max}$ chosen was 0·3 cal cm^{-2} min^{-1} = 432 cal cm^{-2} day^{-1}. This probably overestimates the maximum deviation from the zonal average heating, especially in summer. However, since the magnitude of the perturbation is directly proportional to $S_{\max}$, one can easily infer the disturbance amplitude for other values of $S_{\max}$.

For the characteristic horizontal dimensions of relative heating and cooling one must look again to observations. The separation of primary maxima along a given latitude is approximately the distance between the east coasts of Asia and North America, i.e., 160° longitude. A noteworthy exception is the wintertime-condensation source at the west coast of North America (Fig. 1 (a)). The effect of this source is probably compensated for by the upward motion along the western slopes of the Rockies. Evidence of this lies in the fact that the sea-level normal (see Fig. 6 (a) below) indicates warm-air advection along the west coast.

The wavelength of the principal Fourier component of the heat source-sink distribution in the north-south direction varies from 30° to 60° latitude. Since we are interested only in a general quantitative indication of the nature and magnitude of the disturbances it will suffice for our purposes to consider two extreme values. For computational convenience, these lengths are chosen to be 35° and 53·9° latitude. The latter value is probably closer to the average lateral width.

In order to utilize the values of the confluent hypergeometric functions which are already tabulated, the physical parameters which determine r in (26) were adjusted slightly. For greater accuracy, it was necessary to recalculate Charney's (1947) values of the derivatives.

The quantities which characterize the zonally-averaged stratified current were taken to be : $\bar{T} = 260°K$, $\gamma = 6{\cdot}5°C\ km^{-1}$, $\gamma_d = 9{\cdot}8°C\ km^{-1}$, $\Lambda = 2{\cdot}25\ m\ sec^{-1}\ km^{-1}$, and $U(0) = 2{\cdot}29\ m\ sec^{-1}$. The rigid top was placed at $z_T = 10$ km. We choose $f = 10^{-4}\ sec^{-1}$ and $\beta = 1{\cdot}61 \times 10^{-11}\ m^{-1}\ sec^{-1}$, corresponding to approximately 45° latitude. The two properties of the heating which were kept constant throughout were : $L = 160°$ longitude $= 12{\cdot}61 \times 10^3$ km at 45° latitude and $S_{max} = 0{\cdot}3\ cal\ cm^{-2}\ min^{-1} = 432\ cal\ cm^{-2}\ day^{-1}$.

When $D = 35°$ latitude ($= 3{\cdot}89 \times 10^3$ km), $r = 0{\cdot}500$; when $D = 53{\cdot}9°$ latitude ($= 5{\cdot}99 \times 10^3$ km), $r = 0{\cdot}700$. Hereafter the following notation will be used : $D_1 = D = 35°$ latitude and $D_2 = D = 53{\cdot}9°$ latitude. Calculations were performed for D_1 with $h = 1$, 2, and 5 km, and for D_2 with $h = 2$ km. Two cases were treated : $F = 0$ and $F = 2{\cdot}0 \times 10^{-6}\ sec^{-1}$.

The results with no friction are given in Figs. 2 (a) and 2 (b). In this case $\sigma \to 0$ in (26) and in the limit $E_1 = E_2 = V_2(z) = W_2(z) \equiv 0$, so that the disturbance is in phase or exactly out of phase with respect to height. Figs. 2 (a) and 2 (b), therefore, only show $V_1 \equiv V$ and $W_1 \equiv W$.

In each case V is negative at least below 6 km, thereby implying that in this portion of the troposphere there is a trough one-quarter of a wavelength to the east of a maximum in Q and a ridge the same distance downstream from a minimum in Q. Where the heating is concentrated at lower levels (smaller h), the maximum in V is larger and occurs

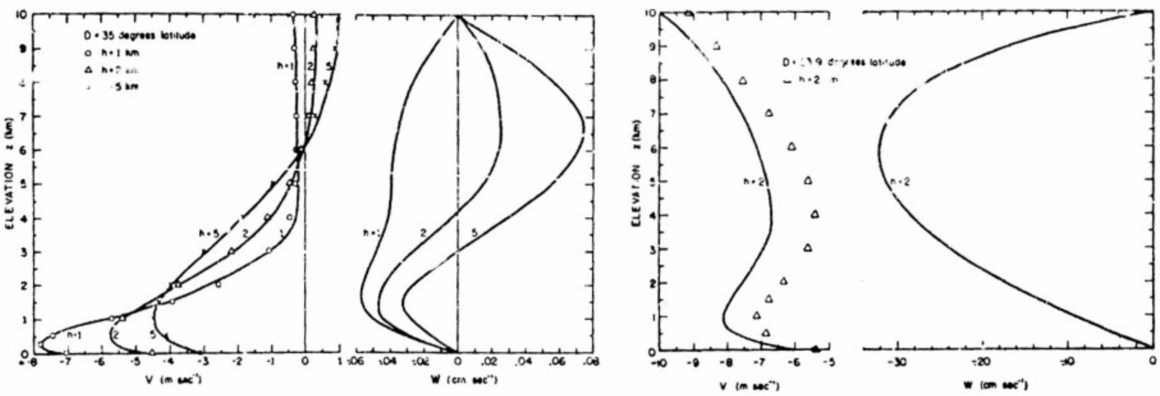

Figure 2 (a). $D = 35°$ latitude ($\equiv D_1$). Figure 2 (b). $D = 53{\cdot}9°$ latitude ($\equiv D_2$).

Figure 2. Vertical distribution of perturbation meridional component of velocity v, and vertical component w at maximum of a heat source for different vertical distributions of heating as given by h. Tropopause taken to be rigid. Friction is not included. $L = 160°$ longitude. Solid lines are computed from general baroclinic model. Circles, triangles and crosses indicate points computed from advective model.

at a lower elevation. For the narrower lateral width, D_1, V becomes small above about 4 km for all values of h. However, for D_2, there is a minimum at this elevation, and then the disturbance increases up to the tropopause. W may be computed from either the thermodynamic energy equation or the vorticity equation. For D_1, the vertical motions are of an order of magnitude less than those usually associated with large-scale daily disturbances, but for D_2, W is of the same order of magnitude as is found in daily motions.

In the lower levels, the vertical motion is *downward* over the heat source. This can be inferred immediately by integrating the vorticity equation (2) from the ground to a level z and making use of (12) and the fact that $V_2(z) = W_2(z) \equiv 0$:

$$W = -\frac{\mu^2 + k^2}{\rho f} \int_0^z \rho V (U - U_c)\, dz \qquad . \qquad . \qquad . \quad (32)$$

Since $U < U_c$ and $V < 0$ in the lower levels, then $W < 0$. The small vertical motions for D_1 must mean that the non-adiabatic heating is almost entirely balanced by the horizontal advection of entropy. This does not appear to be so for D_2.

The results for the model with friction are given in Figs. 3 (a), (b), and (c). (D_1 : $h = 5$ km. has been omitted). In the case of no friction, a change of phase with height of $\frac{1}{2} L$ occurred discontinuously for D_1 : $h = 2$, and not at all for D_1 : $h = 1$ km and for D_2 : $h = 2$ km. With friction the tilt with height is continuous, being rapid in the lower atmosphere and becoming almost zero in the upper levels. Here the phase shift for all the h with $D = D_1$ is approximately $\frac{1}{4} L$ to the west from the ground to 10 km. The phase shift is still less for D_2. The absolute maximum of V in the low levels is damped by friction. As in the case without friction, when the heating is concentrated near the ground a greater disturbance is produced at lower levels, but in contradistinction to the non-frictional case a greater disturbance is also produced in upper levels. Since the magnitude of the surface disturbance determines the frictionally-produced vertical velocity through the top of the friction layer, it also determines the compensating convergence or divergence in the upper atmosphere. Thus, in the case of friction the increased divergences result in increased disturbance amplitudes in the upper levels.

The vertical velocities for D_1 are an order of magnitude larger with than without surface friction, and thereby completely obscure the non-frictional effect. For the case of this narrow lateral width the maximum of downward motion is over the surface ridge and that of upward motion is over the trough. Since the non-frictional vertical velocity for D_2 was large to begin with, the introduction of friction increases the pre-existing maximum of downward motion, which now occurs at a level $1\frac{1}{2}$ km higher than for D_1, and shifts it to a position about $\frac{1}{8} L$ to the east of the surface ridge; the maximum of upward motion is similarly located with respect to the surface trough.

The phase lag between surface trough and maximum Q is now in each case less than a quarter of a wavelength. Since $L = 160°$ longitude, the trough is about 25° longitude to the east of maximum Q. When the heating is concentrated closer to the ground, the rapid change of phase with height occurs at lower levels. For D_1, the 500-mb trough lags to the west of the surface trough by a distance of about 40° longitude and is about 15° to the west of the maximum Q. The lag between surface and 500-mb trough for D_2 is somewhat less, about 25°, so that the 500-mb trough is almost directly over the maximum Q.

From Fig. 3 we see that the perturbation-pressure amplitude at the surface is of the order of 10 mb. The disturbed height of the 500-mb surface for D_1 is approximately 200 ft, but is almost four times as large for D_2. The pressure amplitudes, as well as the velocity components, are of the order of magnitude of the observed disturbances.

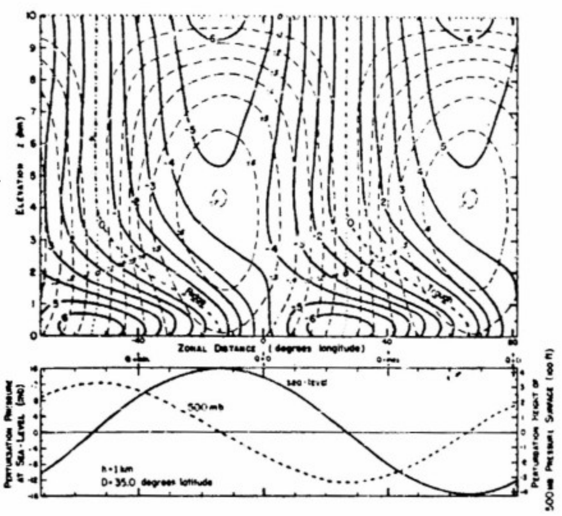

Figure 3 (a)

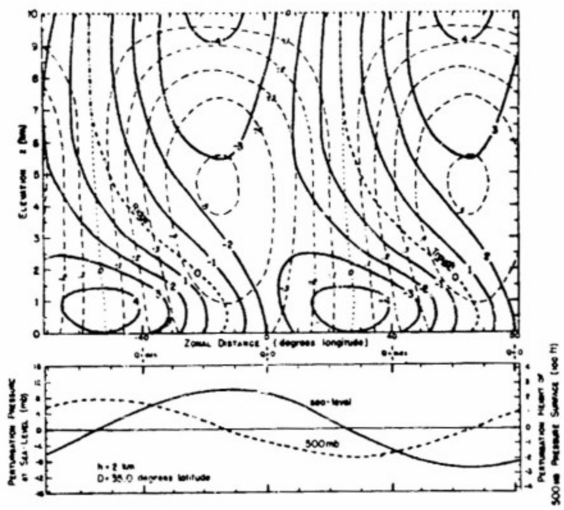

Figure 3 (b)

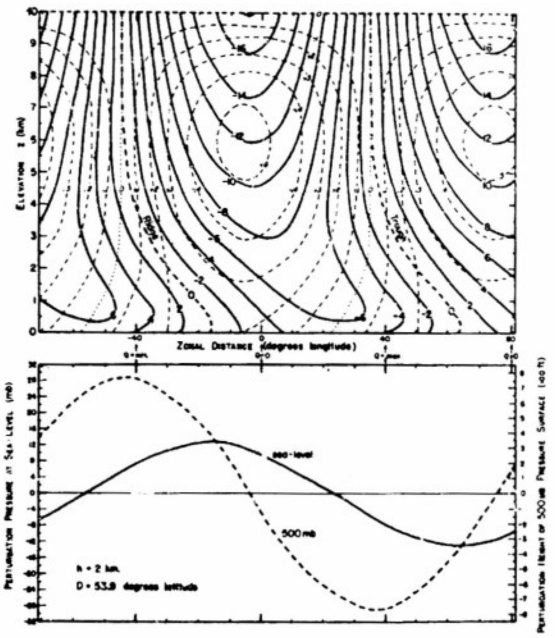

Figure 3 (c)

Figure 3. General baroclinic model with surface friction, at latitude of maximum heating and cooling. Tropopause taken to be rigid. Upper part : zonal section of perturbation meridional velocity v (solid lines) in m sec^{-1} and perturbation vertical velocity w (dashed lines) in cm sec^{-1}. Lower part : zonal profile of perturbation pressure at sea level (solid lines) and perturbation height of 500-mb pressure surface (dashed lines).

5. The advective approximation

It is interesting to see to what extent a simpler baroclinic model is capable of determining the disturbed motions. We make the *a priori* assumption that in the steady state (3) may be written :

$$v_h \cdot \nabla_h \ln \theta \simeq Q \qquad (33)$$

which is tantamount to assuming that the static stability has little influence on the motions. This advective approximation has been used by Sutcliffe (1947) in the investigation of daily motions. With this approximation, the use of the confluent hypergeometric function can be circumvented. The results of the general baroclinic calculations show that the above approximation is not valid in the case of friction since the vertical velocities are large. The advective model will therefore not be applied to the cases with friction in the following. In each case we will test the validity of (33) *a posteriori*.

With the above approximation and with $V_2 = 0$ and $V_1 \equiv V$ in (12) we have

$$\frac{1}{U^2}\left(U\frac{dV}{dz} - \Lambda V\right) = \frac{d}{dz}\left(\frac{V}{U}\right) = \frac{g\chi}{fU^2} \quad . \quad . \quad . \quad (34)$$

The vorticity equation, namely the unintegrated form of (32), is

$$\frac{d\,(\rho W)}{dz} = -\left(\frac{\mu^2 + k^2}{f}\right)(U - U_c)\,\rho V \quad . \quad . \quad . \quad (35)$$

We now follow the procedure outlined by Charney, Fjörtoft and von Neumann (1950). (34) may be integrated from $z = 0$ to z :

$$\frac{V(z)}{U(z)} - \frac{V(0)}{U(0)} = \frac{g}{f}\int_0^z \chi U^{-2}\,dz \equiv -Y(z) \quad . \quad . \quad . \quad (36)$$

We define the vertical pressure mean of a quantity :

$$\overline{\Xi} \equiv \frac{1}{p\,(z_T) - p\,(0)}\int_{p(0)}^{p(z_T)} \Xi\,dp = \frac{g}{p\,(0) - p\,(z_T)}\int_0^{z_T} \Xi\,\rho\,dz \quad . \quad (37)$$

The mean of the vorticity equation (35) subject to the boundary conditions $\rho W = 0$ at $z = 0,\ z_T$ is

$$\overline{UV} = U_c\,\overline{V} \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

But from (36) we have

$$\overline{UV} = \frac{V(0)}{U(0)}\,\overline{U^2} - \overline{U^2\,Y} \quad . \quad . \quad . \quad . \quad (39)$$

and also

$$\frac{V(0)}{U(0)} = \frac{\overline{V}}{\overline{U}} + \frac{\overline{UY}}{\overline{U}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

We may now eliminate $\overline{UV}$ and $V(0)/U(0)$ between (38), (39), and (40), thereby obtaining an expression for V in terms of the given quantities U, Y, and U_c :

$$\overline{V} = \frac{\overline{U^2\,Y} - \overline{U^2}\;\overline{U}^{-1}\;\overline{UY}}{\overline{U^2}\;\overline{U}^{-1} - U_c} \quad . \quad . \quad . \quad . \quad (41)$$

At any level V can then be found by eliminating $V(0)/U(0)$ between (36) and (40) :

$$V = U\,\overline{U}^{-1}\,(\overline{V} + \overline{UY} - \overline{U}Y) \quad . \quad . \quad . \quad (42)$$

V was computed by performing the indicated quadratures numerically using the same mean properties of the flow as before. The results are indicated in Fig. 2 (a) for $h = 1$, 2, and 5 km and in Fig. 2 (b) for $h = 2$ km. For D_1, where the vertical velocities are small, the deviations from the results of the general baroclinic model are within the error of numerical integration. On the other hand, the larger vertical motions for $D = D_2$ produce appreciable deviations in the magnitude of V, although the vertical variation is quite similar.

From (41) we see that the perturbations can become infinite when $U_c = \overline{U^2}/\overline{U}$, that is, when the wavelength of the heating distribution is equal to the stationary wavelength of the baroclinic flow. Resonance can also occur in the general baroclinic model without friction since it too is a linear system; however, this phenomenon is avoided by introducing the dissipative effect of surface friction into the model.

6. The influence of the stratosphere

We shall now proceed to determine the tropospheric motions under the influence of an upper layer of moderate static stability. Assume the tropopause to be a material surface. Let us require that the pressure be continuous everywhere. In particular consider the pressure on either side of the tropopause. Since this is a stream surface by definition, then by combining the dynamic and kinematic boundary conditions we have

$$\frac{d^{\mathrm{I}}(p^{\mathrm{I}} - p^{\mathrm{II}})}{dt} = \frac{d^{\mathrm{II}}(p^{\mathrm{I}} - p^{\mathrm{II}})}{dt} = 0 \quad . \quad . \quad . \quad (43)$$

The superscripts I and II refer to the tropospheric and stratospheric sides, respectively. The pressure on either side of the tropopause is written as a mean and a small perturbation, $P^{\mathrm{I}} + p^{\mathrm{I}}$ or $P^{\mathrm{II}} + p^{\mathrm{II}}$. The boundary conditions (43) for stationary perturbations on a zonal current then become :

$$U^{\mathrm{I,\,II}} \frac{\partial (p^{\mathrm{I}} - p^{\mathrm{II}})}{\partial x} + v^{\mathrm{I,\,II}} \frac{\partial (P^{\mathrm{I}} - P^{\mathrm{II}})}{\partial y} + w^{\mathrm{I,\,II}} \frac{\partial (P^{\mathrm{I}} - P^{\mathrm{II}})}{\partial z} = 0 \quad . \quad (44)$$

Here the quantities are measured at the mean tropopause, but this approximation results in an error of only second order. We impose the quasi-static and quasi-geostrophic approximations on the mean and perturbed motions :

$$fU^{\mathrm{I,\,II}}(\rho^{\mathrm{I}} v^{\mathrm{I}} - \rho^{\mathrm{II}} v^{\mathrm{II}}) - fv^{\mathrm{I,\,II}}(\rho^{\mathrm{I}} U^{\mathrm{I}} - \rho^{\mathrm{II}} U^{\mathrm{II}}) - gw^{\mathrm{I,\,II}}(\rho^{\mathrm{I}} - \rho^{\mathrm{II}}) = 0 \quad . \quad (45)$$

where ρ is a mean quantity. Upon requiring that the mean zonal velocity and the mean density be continuous through the material surface, we have that

$$v^{\mathrm{I}} = v^{\mathrm{II}} \quad . \quad . \quad . \quad . \quad . \quad (46)$$

Therefore the perturbation meridional component is continuous, although $\partial U/\partial z$ or $\partial \rho/\partial z$ may undergo a jump through the surface. Let us define the tropopause as the surface

$$F(x, y, z) = z - Z(y) - \eta(x, y) = 0, \ \eta^2 \ll Z^2 \quad . \quad . \quad (47)$$

By the kinematic boundary condition, the normal velocity on either side must vanish. Thus for steady motions

$$\mathbf{v}^{\mathrm{I}} \cdot \nabla F = \mathbf{v}^{\mathrm{II}} \cdot \nabla F \quad . \quad . \quad . \quad . \quad (48)$$

so that

$$U^{\mathrm{I,\,II}} \frac{\partial \eta}{\partial x} + v^{\mathrm{I,\,II}} \frac{dZ}{dy} - w^{\mathrm{I,\,II}} = 0 \quad . \quad . \quad . \quad (49)$$

Again with $U^{\mathrm{I}} = U^{\mathrm{II}}$ we have

$$w^{\mathrm{I}} - w^{\mathrm{II}} = (v^{\mathrm{I}} - v^{\mathrm{II}}) \frac{dZ}{dy} \quad . \quad . \quad . \quad . \quad (50)$$

which relates the jump in w to that of v. But since we have shown v to be continuous then w also must be continuous :

$$w^{\mathrm{I}} = w^{\mathrm{II}} \quad . \quad . \quad . \quad . \quad . \quad (51)$$

Assume that the zonal mean temperature and velocity are constant with height in the stratosphere and are continuous at the tropopause. One can then show from the thermal-wind relation (cf., Charney 1947) that the mean tropopause must be isothermal and have a slope given by

$$\frac{dZ}{dy} \simeq \frac{fA}{\gamma g} T(z_T) \quad . \quad . \quad . \quad . \quad . \quad (52)$$

Furthermore the zonal velocity in the stratosphere has a meridional variation given by

$$\frac{dU^{II}}{dy} \simeq -\frac{f\Lambda^2}{\gamma g} T(z_T) \quad . \quad . \quad . \quad . \quad (53)$$

For a variable U^{II}, the left side of the steady-state perturbation form of the vorticity equation (1) becomes

$$U^{II} \frac{\partial \zeta^{II}}{\partial x} + \left(\beta - \frac{d^2 U^{II}}{dy^2}\right) v^{II} + \left(f - \frac{dU^{II}}{dy}\right) \nabla_h \cdot \mathbf{v}^{II}$$

However, it can be shown for the values of the mean properties of our model that $\beta \gg d^2 U^{II}/dy^2$ and $f \gg d\, U^{II}/dy$. With this approximation and taking $\chi = 0$ in the stratosphere, Eqs. (13) become for V_1^{II} and V_2^{II} :

$$\frac{d^2 V_{1,2}^{II}}{dz^2} - \frac{1}{H^{II}} \frac{dV_{1,2}^{II}}{dz} - a^{II^2} \left(\frac{U^{II} - U_c}{U^{II}}\right) V_{1,2}^{II} = 0 \quad . \quad . \quad (54)$$

where

$$\left.\begin{aligned} &U_c = U_c^{*II}, \quad U^{II} = U^I(z_T) \\ &a^{II^2} = (\mu^2 + k^2)\, \nu^{II^2}/f^2, \quad \nu^{II^2} = g\gamma_d/\bar{T}^{II} \\ &H^{II} = R\bar{T}^{II}/g, \quad \bar{T}^{II} = T^{II} = T^I(z_T) \end{aligned}\right\} \quad . \quad . \quad (55)$$

Since the coefficients are constant, if we require that $v \to 0$ as $z \to \infty$, the solutions for V_1^{II} and V_2^{II} are :

$$\left.\begin{aligned} &V_{1,2}^{II} = M\, e^{mz} \\ &m = \frac{1}{2H^{II}} - \left(\frac{1}{4H^{II^2}} + a^{II^2} \frac{U^{II} - U_c}{U^{II}}\right)^{\frac{1}{2}} < 0 \end{aligned}\right\} \quad . \quad . \quad (56)$$

The constant M is determined by the condition of continuity at the tropopause : $V_{1,2}^{I} = V_{1,2}^{II}$.

By considering the thermodynamic energy equation on either side of the tropopause and requiring continuity in U, ν and w, we have

$$\frac{\partial v^I}{\partial z} - \left(\frac{\nu^I}{\nu^{II}}\right)^2 \frac{\partial v^{II}}{\partial z} = \frac{\Lambda v}{U}, \; z = z_T \quad . \quad . \quad . \quad . \quad (57)$$

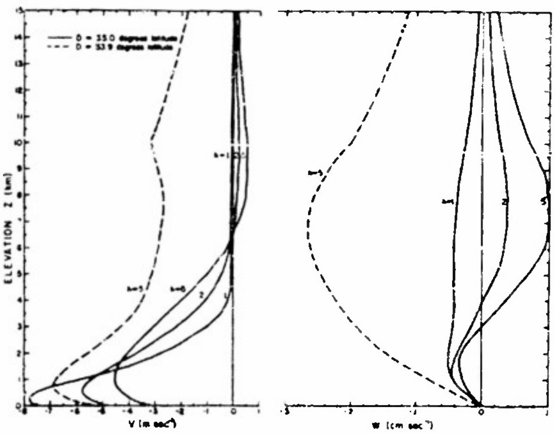

Figure 4. Vertical distribution of perturbation meridional component of velocity v, and vertical component w at maximum of a heat source for different vertical distributions of heating given by h. Stable stratosphere taken into account. Friction is not included. $L = 160°$ longitude. Solid lines $D = 35°$ latitude ($\equiv D_1$). Dashed lines $D = 53{\cdot}9°$ latitude ($\equiv D_2$).

With (12) we have for V_1 and for V_2

$$\frac{1}{V_{1,2}{}^{\mathrm{I}}}\frac{dV_{1,2}{}^{\mathrm{I}}}{dz}-\left(\frac{\nu^{\mathrm{I}}}{\nu^{\mathrm{II}}}\right)^2\frac{1}{V^{\mathrm{II}}}\frac{dV_{1,2}{}^{\mathrm{II}}}{dz}=\frac{\Lambda}{\bar{U}},\ z=z_T \qquad (58)$$

Using Eqs. (19) and (56) for the lower and upper fluids, respectively, we have for ϕ_1 and ϕ_2 that

$$\frac{\phi_{1,2}{}'(\xi_T)}{\phi_{1,2}(\xi_T)}=\lambda(\xi_T)+\frac{m}{2a^{*\mathrm{I}}}\left(\frac{\nu^{\mathrm{I}}}{\nu^{\mathrm{II}}}\right)^2\equiv\lambda^*(\xi_T) \qquad (59)$$

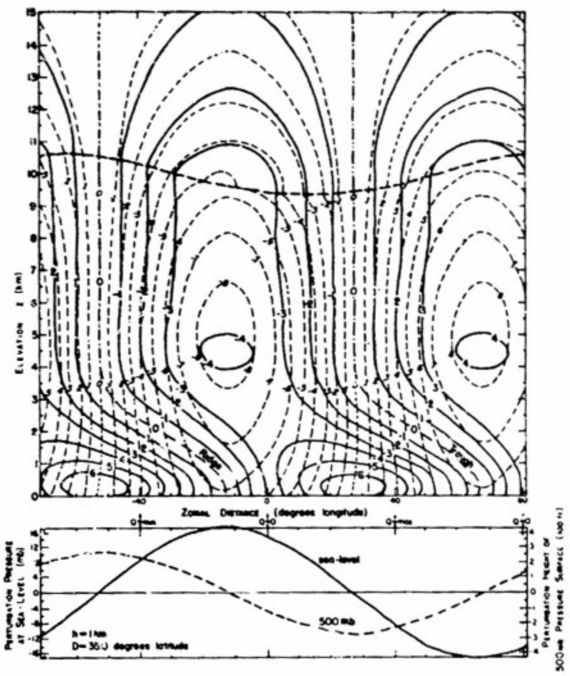

Figure 5 (a)

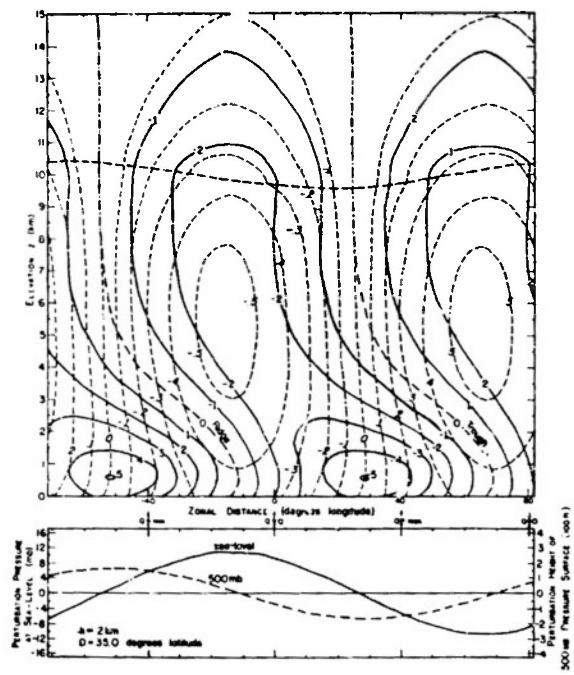

Figure 5 (b)

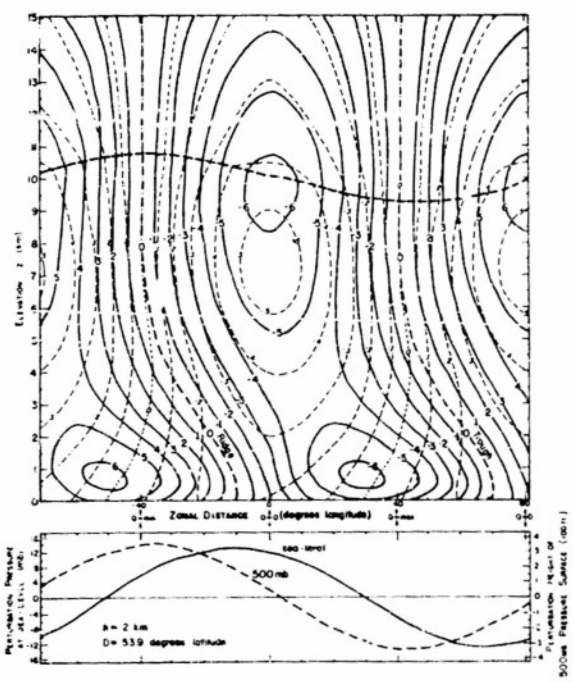

Figure 5 (c)

Figure 5. General baroclinic model with surface friction, at latitude of maximum heating and cooling. Stable stratosphere taken into account. Upper part : zonal section of perturbation meridional velocity v (solid lines) in m sec^{-1} and perturbation vertical velocity w (dashed lines) in cm sec^{-1}. Lower part : zonal profile of perturbation pressure at sea level (solid lines) and perturbation height of 500-mb pressure surface (dashed lines).

where $\lambda(\xi_T)$ has been defined in (23) :

$$\lambda(\xi_T) = \tfrac{1}{2} - \frac{1}{4a^{*\mathrm{I}} H^{*\mathrm{I}}} + \frac{1}{\xi_T} \quad \text{and} \quad \xi = 2\,a^{*\mathrm{I}}\, U^{\mathrm{I}}/\Lambda \quad . \quad . \quad (60)$$

By the condition of continuity of the velocity components at the tropopause and the thermodynamic energy equation for the stratosphere we have upon using (56) :

$$W^{\mathrm{I}} = W^{\mathrm{II}} = -fm\, U'V^{\mathrm{I}}\,(\nu^{\mathrm{II}})^{-2} \quad \text{at} \quad z = z_T \quad . \quad . \quad . \quad (61)$$

The solutions for this model may be found in exactly the same manner as for the model with the rigid top – we need only to replace condition (22) by (59). The results in absence of friction are given in Fig. 4 and those with surface friction are given in Figs. 5 (a), (b), and (c).

7. Resonance

For the cases of the narrow lateral extent ($D = D_1$: $\mu^2 + k^2 = 2{\cdot}86 \times 10^{-6}\,\mathrm{km}^{-2}$) we find that when surface friction is taken into account the influence of the stratosphere on the tropospheric motions is negligible below 6 km while the amplitudes in the upper troposphere are reduced by no more than a factor of two. On the other hand, for the case with large lateral width ($D = D_2$: $\mu^2 + k^2 = 1{\cdot}35 \times 10^{-6}\,\mathrm{km}^{-2}$) the reduction at the tropopause is by a factor of three. The explanation lies in the fact that the latter case is closer to the non-frictional resonant wave number. Although infinite amplitudes are prevented by the inclusion of dissipative forces, large but bounded disturbances may still occur in the near vicinity of the non-frictional resonant wave number. We have shown that in the advective model with a rigid top, resonance occurs when $U_c = \overline{U^2}/\overline{U}$, that, is when $\mu^2 + k^2 = 1{\cdot}08 \times 10^{-6}\,\mathrm{km}^{-2}$. We can also determine the critical wave number for the general baroclinic model with little difficulty. Considering the limit when $\sigma \to 0$ in Eqs. (26), we find that

$$\left.\begin{aligned} E_1 &= E_2 = 0 \\ B(\xi_0) &= -\frac{C_3\,\kappa(\xi_0) - C_1\left[\kappa(\xi_T) + C_4 C_6 - C_3 C_5\right]}{C_1 C_4 - C_2 C_3} \\ A(\xi_0) &= \frac{C_4\,\kappa(\xi_0) - C_2\left[\kappa(\xi_T) + C_4 C_6 - C_3 C_5\right]}{C_1 C_4 - C_2 C_3} \end{aligned}\right\} \quad . \quad . \quad (62)$$

Thus $A(\xi_0)$ and $B(\xi_0)$ become infinite when the denominator vanishes, i.e., when $C_1 C_4 = C_2 C_3$. This condition may be written in terms of the confluent hypergeometric functions with the aid of (34) :

$$\frac{[\psi_1'(\xi_T)\psi_2'(\xi_0) - \psi_1'(\xi_0)\psi_2'(\xi_T)] - \lambda(\xi_0)[\psi_1'(\xi_T)\psi_2(\xi_0) - \psi_2'(\xi_T)\psi_1(\xi_0)]}{[\psi_2'(\xi_0)\,\psi_1(\xi_T) - \psi_1'(\xi_0)\psi_2(\xi_T)] - \lambda(\xi_0)[\psi_1(\xi_T)\psi_2(\xi_0) - \psi_1(\xi_0)\psi_2(\xi_T)]} = \lambda(\xi_T) \quad . \quad (63)$$

For the model with a stratosphere one need only replace the right side of (63) with $\lambda^*(\xi_T)$. Using Charney's tables (1947), the left- and right-hand sides may be plotted as functions of r. The intersection indicates the value of r for which the above equation is satisfied. The result is that resonance occurs at $\mu^2 + k^2 = 1{\cdot}22 \times 10^{-6}\,\mathrm{km}^{-2}$ with a rigid top and at $1{\cdot}05 \times 10^{-6}\,\mathrm{km}^{-2}$ when there is an isothermal upper layer. This means that the influence of the stratosphere is to remove the model further from resonance than if there were a rigid top, all other properties of the models being the same.

Although the advective model with friction included cannot be expected to give reliable quantitative results, some qualitative features are of interest. One can introduce friction into the advective model in exactly the same manner as into the more general baroclinic model. This is done by using (15) for the lower boundary condition in the vertical integration of (35). Since the mathematical development is straightforward it is

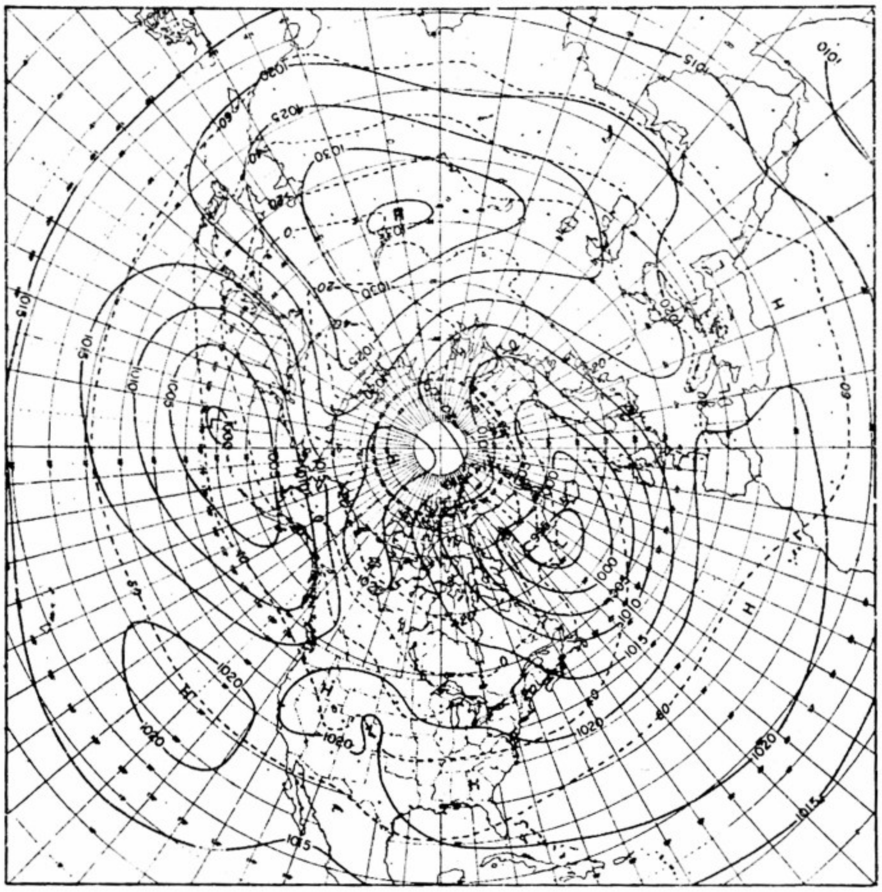

Figure 6 (a). January.

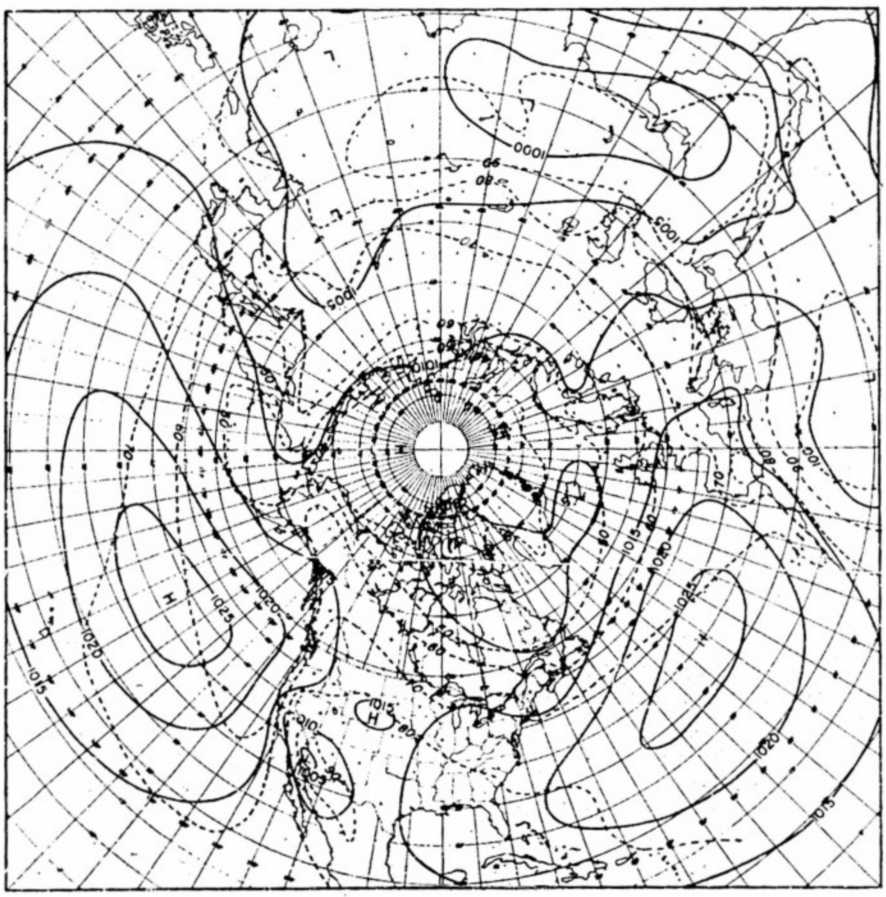

Figure 6 (b). July.

Figure 6. Normal monthly sea-level pressure (solid lines) in mb after United States Weather Bureau (1952) and temperature (dashed lines) in °F after Haurwitz and Austin (1944).

not reproduced here. By the condition for resonance for the advective model one sees that resonance may be reached either by reducing $\mu^2 + k^2$ or by reducing the zonal velocity. When this occurs the character of the stationary disturbance changes markedly. For instance, when $\mu^2 + k^2$ is taken to be $0{\cdot}90 \times 10^{-6}$ km^{-2} (for $L = 160°$ longitude this corresponds to $D \simeq 70°$ latitude), $U(z)$ is kept the same as before, and $h = 2$, the sea-level trough is located about 30°W of the heating minimum (cold source), and slopes *eastward* with height, so that at 500 mb the trough is 30° E of the heating minimum. The sea-level ridge is about 50°E of the heating minimum and at 500 mb is 110°E of the minimum*.

8. The observed normals in the light of the theoretical results

In testing the validity of the model on the basis of observation, one can compare only the orders of magnitude and general spatial distribution of the disturbances in relation to the field of heating. In the discussion which follows we shall refer to the maps of normal seasonal heating (Figs. 1 (a) and (b)) and to the maps of normal sea-level pressure and temperature (Figs. 6 (a) and (b)). The sea-level pressure is taken from tentative copies of new normals compiled by the United States Weather Bureau (1952) which were kindly made available before their publication. The normal sea-level temperatures were taken from the book *Climatology* by Haurwitz and Austin (1944). Although the 500-mb normals are not shown here, the reader is referred to the previously mentioned Weather Bureau publication. In order to facilitate the discussion, zonal profiles of the sea-level pressure and the height of the 500-mb surface (Fig. 7) are included. These are presented as deviations from the zonal average in a belt 20° latitude wide centred at 45°N in winter and at 50°N in summer. The continental elevations are also indicated schematically. The information on non-adiabatic heating contained in Fig. 1 leaves much to be desired regarding the location of the primary sources and sinks over the continents. However, one may deduce their locations from the geostrophic temperature advection at the surface provided there are no large changes in the continental elevations. The locations of primary extremes of heating determined from Fig. 1 and, where permissible, from surface temperature advection are also shown schematically in Fig. 7.

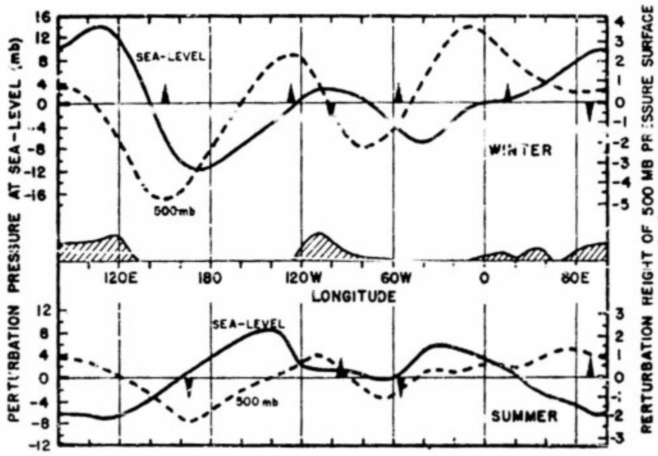

Figure 7. Zonal profiles of normal perturbation pressure at sea level (solid line) and normal perturbation height of the 500-mb pressure surface (dashed line). Continental elevations are indicated schematically by cross-hatched area. Positions of observed relative heat sources and sinks indicated by spires pointed upward and downward respectively. Upper part : winter profiles averaged over 20° latitude centred at 45°N. Lower part : summer profiles averaged over 20° latitude centred at 50°N.

* This is in substantial agreement with results obtained independently by B. Gilchrist who has made a similar investigation using a '2½ dimensional' baroclinic model. These results were communicated privately to the writer by Dr. Gilchrist who is at present at the Institute for Advanced Study.

(a) *Winter*

At 500 mb, the January normal exhibits an extremely broad trough with a primary minimum at the longitude of the western Great Lakes and a slight indication of a secondary minimum approximately at Newfoundland. The existence of the secondary trough is not apparent from the averaged profile in Fig. 7. The western extension is quasi-barotropic in structure : this trough is in phase at all levels in the troposphere but has its minimum amplitude near the surface. On the other hand, the eastern component increases in intensity and shifts eastward as we go to lower elevations. At the surface this eastward extension is observed as the Icelandic low, which is far more intense (a perturbation amplitude of about 7 mb) than the surface trough at the western Great Lakes. The position and intensity of the western component is in agreement with the calculated position and intensity of the orographically produced barotropic trough (cf., Charney and Eliassen (1949) and Bolin (1950)), while the eastern extension appears to be a baroclinic trough generated by sensible heating off the North American east coast and by heat of condensation just inland. These sources are characterized by the narrower lateral width, D_1. The model with friction has a warm sea-level trough about 25°E of the heating maximum with perturbation amplitude of the order of 10 mb. At 500 mb, the computed trough is about 10°W of the position of maximum heating. This is a little further west than the eastern extension of the observed normal trough. The existence of a ridge at 500 mb in the eastern Atlantic has been explained by the calculations of Charney and Eliassen as part of a resonance wave initiated by the Rockies.

In an earlier section it was shown that the heating from the pronounced condensation maximum along the North American west coast north of 40°N is over-compensated by the adiabatic cooling caused by the forced ascent along the western slopes of the Rockies. Similarly the large wintertime sink over North America due to downward eddy conduction and nocturnal radiation is completely nullified by the heating due to the forced descent along the eastern slopes. This is verified by the observation that there is weak cold-air advection east of the Rockies. This may explain why the sea-level anticyclone is located in the extreme western part of the continent, to the west of the non-adiabatic cold source, and is rather poorly developed compared with the Asiatic high-pressure cell.

In the far western Pacific the position of maximum precipitation almost coincides with that of the maximum upward eddy flux of sensible heat from the warm Kuroshio current. The sea-level Aleutian low appears to have the correct location with respect to these heat sources. As predicted by theory it lies roughly 25°E of the heat sources. The Siberian surface high-pressure cell is without doubt generated by the cold source to the west, the location of which was deduced in Section 1 from surface warm air advection.

The calculations indicate that the lower the level of the sources or sinks the more rapidly the phase of the disturbance changes with height. Off east coasts in winter the sources are due to a combination of condensation and eddy conduction and thus have a maximum at about 2 km. On the other hand the wintertime sink in western Asia, which is due to downward eddy conduction and nocturnal radiation, occurs at a lower level. The change of phase with height of the Aleutian low is consequently much less rapid than that of the Siberian anticyclone.

A further result of theory is that for equal amounts of heating in a vertical column, low-level sources or sinks produce a larger surface disturbance than sources and sinks at moderate levels. This would imply that the total relative cooling over central Siberia could be of appreciably smaller magnitude than the relative heating in the western Pacific and yet produce a surface monsoonal high of the same relative magnitude as the low off the east Asiatic coast.

The condensation source in the Mediterranean Sea has dimensions which are considerably smaller than those of the sources and sinks which we have hitherto considered. This, added to the fact that the positions of the Mediterranean heat source and of the sea-level pressure minimum coincide, leads one to believe that the controlling mechanism in this case is more of the type that generates large-scale sea-breezes.

(b) *Summer*

We shall now attempt to diagnose the changes observed in the summer normal flow pattern. Since we are interested only in the perturbations of the zonal motion, we must examine them at a higher latitude corresponding to the northward shift of the westerlies. In the summer hemisphere there are no large regions of strong absolute cooling such as are found in the winter over continents. This means that regions of heating of large magnitude in the summer have deviations from the zonal mean heating which are considerably less effective in disturbing the flow. On the other hand, there is a counteracting element. Although the zonal momentum is reduced by more than one-half at the tropopause level from winter to summer, the low-level reduction is not nearly so great. Since the influence of the relative sources and sinks is most affected by the magnitude of the mean velocities, say, in the lowest 3 km, one should expect some increase in the disturbance produced by a given amount of relative heating.

The wintertime maximum of upward flux of sensible heat from the Gulf Stream off the North American east coast (cf. Fig. 7) vanishes in the summertime. The relative minimum of heating in the western Atlantic is in agreement with the warm-air advection found there and its location is by theory consistent with the position of the weak sea-level ridge to the east. There is now an upward flux of sensible heat over the continent, and the precipitation maximum over Eastern United States is broader but appears at a lower latitude. The principal sea-level Atlantic trough is now located over the east coast of North America, and is considerably less intense than the corresponding winter trough (the Icelandic low). This would indicate that the reduction of the low-level zonal velocity in summer is not sufficient to counteract the decrease in the contrast between heating and cooling.

Both the heating theory and the theory of orographical influences predict a trough at 500 mb which is further to the west than the weak trough observed at the North American east coast. No explanation has yet been found for this discrepancy.

The dimensions of the summertime low-pressure area in south-western United States indicate that the motions here are governed by the same mechanism as the wintertime low over the Mediterranean. Although a condensation minimum occurs in this arid region, it is the location of a sensible-heat source in summer. It seems that the traditional explanation for the formation of this 'heat low' is consistent with scale considerations.

The maximum warm-air advection just off the Asiatic east coast indicates a relative cold source. The wintertime sea-level Aleutian low has virtually disappeared and a trough now exists at about 120°E, approximately 50°W of the relative cold source. The 700-mb and 500-mb normals show that this trough slopes to the east with increasing height and at 500 mb appears at the longitude of the cold source. One would expect a weak 500-mb trough induced by the flow of the summer westerlies over the relatively low mountains to the north of the Tibetan Plateau to appear just to the west of this longitude (Bolin 1950). The advective model with friction yields an eastward slope of pressure axes with height if either $\mu^2 + k^2$ or the zonal velocity is reduced sufficiently. Apparently this has occurred over the Pacific in summer. According to the advective model, the 500-mb thermally-produced trough should be located to the east of the longitude of maximum relative cooling. It thus appears that although the theoretical thermal and orographical disturbances are weak and somewhat out of phase with each other at 500 mb, together they can explain the observed trough. Furthermore, the position of the sea-level ridge in the eastern Pacific relative to the cold source in the western Pacific is also in general agreement with the theoretical results, providing that resonance has been crossed over.

The strong winter anticyclone over eastern Asia is replaced by a general low-pressure area. This is probably a consequence of the reversal of the direction of eddy heat flux and the elimination of long-wave radiational cooling in the lowest layers. The sink has been replaced by a source which, according to our theory, produces a low to the east at sea level.

(c) *Conclusions*

It appears that the mechanism proposed is capable of accounting for the gross features of the observed sea-level normals at middle and high latitudes. Further, the influence of large-scale asymmetries of non-adiabatic heating and cooling on mid-tropospheric flow is calculated to be of the same magnitude as the influence of broad mountain areas. Charney and Eliassen (1949) and Bolin (1950) have concluded that the orographic influence on 500-mb motions is of primary importance whereas Sutcliffe (1951) in a qualitative argument contends that the thermal influence dominates. It must be said that the present quantitative study does not conclusively support or refute either point of view. If one is willing to discount the near-resonance calculations, the computations do indicate that the thermal effect becomes relatively less important as we go to higher levels.

The inclusion of surface friction appears to be necessary in order to avoid resonance. It also yields more realistic motions. In order to obtain an undistorted estimate of the magnitude of the disturbances in the upper third of the troposphere, it appears that the influence of the stratosphere must be taken into account. The large-scale horizontal eddy transport of heat or momentum does not appear to be a deciding factor in determining the quasi-stationary flow, but may be of primary importance for the transient states.

The idealizations regarding the distribution of heating utilized in this investigation were dictated by the lack of information concerning this quantity. Current knowledge does not warrant the use of the observed field for a more precise analysis of the type carried out by Charney and Eliassen for orographically produced perturbations. It is felt, therefore, that with the availability of better data, this aspect of the problem should be pursued further.

Since the normal three-dimensional mass field is known empirically better than the heating field, a companion study to the present one suggests itself. It is possible to test the proposed heating model by deducing the composite three-dimensional field of non-adiabatic heating dynamically from the observed normal maps by means of the non-linear equations. The vorticity equation

$$\mathbf{v}_h \cdot \nabla_h (\zeta + f) = \frac{f}{\rho} \frac{\partial (\rho w)}{\partial z}$$

may be applied to the normals if the eddy stresses and the time-dependent term may be ignored. For a particular monthly normal one can compute the three-dimensional distribution of the horizontal advection of absolute vorticity geostrophically. From this distribution one may determine the difference between the vertical momentum at any level z and the top of the friction layer z_0, thus

$$\rho w - (\rho w)_{z=z_0} = -\frac{1}{f} \int_{z_0}^{z} \rho \mathbf{v}_h \cdot \nabla_h (\zeta + f)\, dz$$

A knowledge of the large-scale orographical features and an estimate of the vertical mean geostrophic wind in the friction layer enables one to calculate the portion of $(\rho w)_{z=z_0}$ resulting from continental elevations. That part of $(\rho w)_{z=z_0}$ due to frictional effects may be estimated by means of some simple frictional law such as the one used in this paper. One now has adequate information to deduce the three-dimensional field of vertical motion dynamically. The large-scale distribution of Q is determined from the steady-state version of the thermodynamic energy equation (3), since now all the quantities besides w can be measured directly from the normals. One of the virtues of such an approach is that the orographic influence on the normals is implicitly taken into account.

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References

Bleeker, W. 1950 *Cent. Proc. R. Met. Soc.*, p. 67.

Bolin, B. 1950 *Tellus*, **2**, p. 184.

Brunt, D. 1939 *Physical and Dynamical Meteorology*, Cambridge University Press.

Charney, J. G. 1947 *J. Met.*, **4**, p. 135.

1948 *Geofys. Publ.*, Oslo, **17**, No. 2.

Charney, J. G. and Eliassen, A. 1949 *Tellus*, **1**, p. 38.

Charney, J. G., Fjörtoft, R. and von Neumann, J. 1950 *Ibid.*, **2**, p. 237.

Eady, E. T. and Sawyer, J. S. 1951 *Quart. J. R. Met. Soc.*, **77**, p. 549.

Elliott, T. D. and Smith, T. B. 1949 *J. Met.*, **6**, p. 67.

Elsasser, W. 1942 *Harvard Met. Studies*, **6**.

Hadley, G. 1735 *Phil. Trans.*, **29**, p. 58.

Halley, E. 1686 *Ibid.*, **26**, p. 153.

Haurwitz, B. 1940 *J. Mar. Res.*, New Haven, **3**, p. 35.

1947 *J. Met.*, **4**, p. 1.

Haurwitz, B. and Austin, J. M. 1944 *Climatology*, New York (McGraw-Hill).

Jacobs, W. C. 1943 *Ann. New York Acad. Sci.*, **44**, p. 19.

1949 *J. Met.*, **6**, p. 266.

1951 *Bull. Scripps Inst. Ocean*, La Jolla, Calif., **6**, p. 27.

Landsberg, H. 1945 *Handbook of Meteorology*, Eds. F. A. Berry, E. Bollay, N. R. Beers, New York (McGraw-Hill), p. 927.

London, J. 1952 *J. Met.*, **9**, p. 145.

Malkus, J. S. and Stern, M. E. 1951 'The flow of a stable atmosphere over a heated island,' *Res. Rep. Phys. Ill. Inst. Tech.*

Möller, F. 1950a *Experientia*, Basel, **6**, p. 361.

1950b *Met. Rund.*, Heidelberg, **3**, p. 97.

Namias, J. 1948 *Trans. Amer. Geophys. Un.*, **29**, p. 777.

1952 *Bull. Amer. Met. Soc.*, **33**, p. 279.

Panofsky, H. A. 1947 *J. Met.*, **4**, p. 35.

Queney, P. 1947 *Misc. Rep. Met. Univ. Chicago*, **23**.

1948 *Bull. Amer. Met. Soc.*, **29**, p. 16.

Rossby, C.-G. 1939 *J. Mar. Res.*, New Haven, **2**, p. 38.

1949 *The atmospheres of the earth and planets*, Ed. G. P. Kuiper, University Chicago Press, p. 16.

Scorer, R. S. 1949 *Quart. J. R. Met. Soc.*, **75**, p. 41.

Starr, V. P. 1951 *Ibid.*, **77**, p. 44.

Stern, M. E. and Malkus, J. S. 1952 *Mar. Met. Rep. Woods Hole Ocean. Inst.*, No. 52/27.

Sutcliffe, R. C. 1947 *Quart. J. R. Met. Soc.*, **73**, p. 370.

1951 *Ibid.*, **77**, p. 435.

United States Weather Bureau 1952 *Normal Weather Charts for the Northern Hemisphere*, U. S. W. B., Tech. Paper No. 21.

Wexler, H. 1936 *Mon. Weath. Rev.*, U. S. W. B., **64**, p. 122.

Willett, H. C. 1944 *Descriptive Meteorology*, New York (Academic Press), Chap. VII.

Wippermann, F. 1951 *Arch. Met. Geophys. Biokl. A*, **4**, p. 112.