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ON THE NUMERICAL INTEGRATION OF THE PRIMITIVE EQUATIONS OF MOTION FOR BAROCLINIC FLOW IN A CLOSED REGION

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ABSTRACT

This paper considers the problem of numerically integrating the primitive equations corresponding to a 2-level model of the atmosphere bounded by two zonal walls on a spherical earth. Inertio-gravitational motions of the external type are filtered *a priori;* for such a constraint it is possible to define a stream function corresponding to the vertically integrated motions. A system of integration is developed for initial conditions which specify the shear wind vector, the specific volume, and the vorticity of the vertically integrated flow. Methods for reducing truncation error and for increasing the rate of convergence of the elliptic part are discussed.

The question of boundary conditions is discussed at length. It is shown that the usual central difference methods yield independent solutions at alternate points, thus providing a source of computational instability to which the primitive equations are particularly sensitive. The solutions may be made compatible by suitable computational boundary conditions which can be deduced as sufficient conditions for insuring that the numerical solutions possess exact integrals. The application of these considerations to viscous flow is also discussed.

1. INTRODUCTION

The object of this paper is to discuss some of the problems of employing the primitive equations as a framework in which to study large-scale atmospheric processes. These problems are to a large extent connected with the deduction of a stable and rational means for numerically integrating the primitive equations. It has been common experience that the application of the primitive equations to large-scale motions has suffered from the delicate balance between the Coriolis and pressure-gradient forces resulting in relatively small accelerations and horizontal divergence. Therefore, attempts to integrate the primitive equations numerically can be successful only if the problem stated in numerical form is properly compatible with the system of continuous (differential) equations. Slight incompatibilities (e.g., incorrect boundary conditions) in systems which are not sensitive in this manner, e.g., those admitting only gravitational motions or only Rossby wave solutions, apparently do not produce a very rapid degeneracy.

The particular system of equations, and the domain of integration to be discussed, has been designed in essence to form the hydrodynamic framework for numerical studies of the dynamics of the general circulation. However, those physical considerations which do not directly or crucially bear on the present objective of establishing a stable mathematical and hydrodynamic framework will be omitted here. At the time of the preparation of this manuscript, stable numerical integrations had been performed over periods in excess of 50 atmosphere days within the context of the system and methods to be described. The physical considerations directly bearing on the construction of this general circulation model, together with the results of the integrations, will form the subject of a later report.

2. NON-LINEAR BAROCLINIC FLOWS

a. DIFFERENTIAL EQUATIONS

The equations of motion in spherical coordinates with height as the vertical coordinate are (see for instance Haurwitz [6]):

$$\frac{r}{m}\ddot{\lambda}+2(\dot{\lambda}+\Omega)\left(\frac{\dot{r}}{m}-r\alpha\dot{\theta}\right)=-\frac{m}{r\rho}\frac{\partial p}{\partial \lambda}+F_\lambda \tag{1a}$$

$$r\ddot{\theta}+2\dot{r}\dot{\theta}+\frac{r\alpha}{m}\dot{\lambda}(\dot{\lambda}+2\Omega)=-\frac{1}{r\rho}\frac{\partial p}{\partial \theta}+F_\theta \tag{1b}$$

$$\ddot{r}-r(\dot{\theta})^2-\frac{r}{m^2}\dot{\lambda}(\dot{\lambda}+2\Omega)=-\frac{1}{\rho}\frac{\partial p}{\partial r}-g+F_r \tag{1c}$$

in which λ is the longitude, positive eastward; θ is the latitude; r is the radial distance from the center of the earth; $m\equiv\sec\theta$; $\alpha\equiv\sin\theta$; Ω is the angular velocity of the earth's rotation; ρ is the density; p is the pressure; g is the acceleration of gravity; $\mathbf{F}=\mathbf{i}F_\lambda+\mathbf{j}\,F_\theta$ is the horizontal component of the frictional force vector, F_r its vertical component, and $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors in the λ and θ directions; and $\dot{(\ )}\equiv d\,(\ )/dt$ is the time change on a material particle. The effect of the ellipsoidal shape of the earth in balancing the centrifugal accelerations has already been taken into account.

The kinetic energy equation for *non-hydrostatic* motions is obtained by multiplying (1a), (1b), and (1c) by $r\dot{\lambda}/m$, $r\dot{\theta}$, and $\dot{r}$, respectively, and adding the resulting equations:

$$\frac{1}{2}\frac{d}{dt}\left[\left(\frac{r\dot{\lambda}}{m}\right)^2+(r\dot{\theta})^2+(\dot{r})^2\right]=-\frac{1}{\rho}\left(\dot{\lambda}\frac{\partial p}{\partial \lambda}+\dot{\theta}\frac{\partial p}{\partial \theta}+\dot{r}\frac{\partial p}{\partial r}\right)$$
$$-g\dot{r}+\frac{r\dot{\lambda}}{m}F_\lambda+r\dot{\theta}F_\theta+\dot{r}F_r. \tag{2a}$$

If we constrain the motions to be quasi-static, then (1c) becomes

$$0=-\frac{1}{\rho}\frac{\partial p}{\partial r}-g \tag{2b}$$

In this case, the individual change of kinetic energy of the horizontal motions calculated from (1a) and (1b) no longer *only* depends on the work done by the pressure gradient and external forces. A correct kinetic energy equation consistent with the quasi-static approximation can be derived if:

(i) the terms $2(\dot{\lambda}+\Omega)\,\dot{r}/m$ and $2\dot{r}\dot{\theta}$ are dropped from (1a) and (1b) respectively, and

(ii) where r appears undifferentiated in (1a) and (1b), it is replaced by a, the mean radius of the earth.

The kinetic energy equation for quasi-static motions then becomes

$$\frac{a^2}{2}\frac{d}{dt}\left[\left(\frac{\dot{\lambda}}{m}\right)^2+(\dot{\theta})^2\right]=-\frac{1}{\rho}\left(\dot{\lambda}\frac{\partial p}{\partial \lambda}+\dot{\theta}\frac{\partial p}{\partial \theta}\right)+a\left(\frac{\dot{\lambda}}{m}F_\lambda+\dot{\theta}F_\theta\right). \tag{2c}$$

Furthermore, the quasi-static assumption permits us to transform the resulting horizontal equations of motion and the hydrostatic equation to a coordinate system in which p is the vertical coordinate (Eliassen [2]):

$$\frac{a}{m}\ddot{\lambda}-2a\alpha(\dot{\lambda}+\Omega)\dot{\theta}=-\frac{m}{a}\frac{\partial \phi}{\partial \lambda}+F_\lambda \tag{3}$$

$$a\ddot{\theta}+\frac{a\alpha}{m}(\dot{\lambda}+2\Omega)\dot{\lambda}=-\frac{1}{a}\frac{\partial \phi}{\partial \theta}+F_\theta \tag{4}$$

$$0=\frac{\partial \phi}{\partial p}+\frac{1}{\rho} \tag{5}$$

where $\phi\equiv g(r-a)$ is the geopotential and

$$\dot{(\ )}\equiv\left[\frac{\partial}{\partial t}+\dot{\lambda}\frac{\partial}{\partial \lambda}+\dot{\theta}\frac{\partial}{\partial \theta}+\omega\frac{\partial}{\partial p}\right](\ );\ \omega\equiv\frac{dp}{dt}.$$

Differentiations with respect to t, λ, and θ are now in a constant pressure surface.

The continuity equation is

$$\mathcal{D}+\frac{\partial \omega}{\partial p}=0 \tag{6}$$

where

$$\mathcal{D}\equiv\frac{\partial \dot{\lambda}}{\partial \lambda}+m\frac{\partial}{\partial \theta}\left(\frac{\dot{\theta}}{m}\right). \tag{7}$$

The thermodynamic energy equation is

$$\left.\begin{aligned}\dot{(\ln \Theta)}&=\frac{1}{c_pT}\dot{q}\\ \text{where}\quad \ln\Theta&=\text{const}+(1-\kappa)\ln p+\ln\left(\frac{1}{\rho}\right)\end{aligned}\right\} \tag{8}$$

and Θ is the potential temperature; T is the temperature; $(1-\kappa)=c_v/c_p$ is the ratio of the specific heat of air at constant volume to that at constant pressure; $\dot{q}$ is the non-adiabatic heat added or removed per unit mass per unit time.

Anticipating our ultimate needs for the numerical integration, we will conformally map the sphere onto a Mercator projection. We denote the map coordinates by x and y, positive in the easterly and northerly directions, respectively. For this projection the map scale factor is $m=\sec\theta$, so that

$$\left.\begin{aligned}dx&=ad\lambda\\ dy&=amd\theta\end{aligned}\right\} \tag{9}$$

and

$$\left.\begin{aligned}x&=a\lambda\\ y&=a\ln\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right).\end{aligned}\right\} \tag{10}$$

It will be convenient to deal with the *map velocity* components:

$$\left.\begin{aligned}\mathbf{V}&=\mathbf{i}u+\mathbf{j}v\\ u&\equiv\dot{x}=a\dot{\lambda}\\ v&\equiv\dot{y}=am\dot{\theta}\end{aligned}\right\} \tag{11}$$

where the earth velocity is $\mathbf{V}/m$. The following scalar and vector transformations will be useful:

$$\left.\begin{aligned}\nabla\beta&=\mathbf{i}\frac{m}{a}\frac{\partial \beta}{\partial \lambda}+\mathbf{j}\frac{1}{a}\frac{\partial \beta}{\partial \theta}=m\left(\mathbf{i}\frac{\partial \beta}{\partial x}+\mathbf{j}\frac{\partial \beta}{\partial y}\right)\\ \nabla^2\beta&=\frac{m^2}{a^2}\frac{\partial^2 \beta}{\partial \lambda^2}+\frac{m}{a^2}\frac{\partial}{\partial \theta}\left(\frac{1}{m}\frac{\partial \beta}{\partial \theta}\right)=m^2\left(\frac{\partial^2 \beta}{\partial x^2}+\frac{\partial^2 \beta}{\partial y^2}\right)\\ \nabla\cdot\mathbf{B}&=\frac{m}{a}\left(\frac{\partial B_\lambda}{\partial \lambda}+\frac{\partial B_\theta/m}{\partial \theta}\right)=m\frac{\partial B_x}{\partial x}+m^2\frac{\partial B_y/m}{\partial y}\\ \text{curl}\,\mathbf{B}&=\frac{m}{a}\left(\frac{\partial B_\theta}{\partial \lambda}-\frac{\partial B_\lambda/m}{\partial \theta}\right)=m\frac{\partial B_y}{\partial x}-m^2\frac{\partial B_x/m}{\partial y}.\end{aligned}\right\} \tag{12}$$

Here $\nabla . \mathbf{B}$ has been defined as a 2-dimensional operator in the x–y plane and curl $\mathbf{B}$ is a scalar which has been defined as the vertical component of the usual $\nabla \times \mathbf{B}$ vector.

Equations (3), (4), and (7) become upon being mapped:

$$\dot{u}-2\alpha\left(\frac{u}{a}+\Omega\right)v=-m^2\frac{\partial\phi}{\partial x}+mF_x \tag{13}$$

$$m\left(\frac{\dot{v}}{m}\right)+\alpha\left(\frac{u}{a}+2\Omega\right)u=-m^2\frac{\partial\phi}{\partial y}+mF_y \tag{14}$$

$$\mathcal{D}=\nabla\cdot\left(\frac{\mathbf{V}}{m}\right)\equiv\frac{\partial u}{\partial x}+m^2\frac{\partial v/m^2}{\partial y} \tag{15}$$

where

$$\dot{(\;)}=\left[\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+\omega\frac{\partial}{\partial p}\right](\;). \tag{16}$$

We now construct a 2-level model in the fashion of Eliassen [4]. The atmosphere is divided in the vertical into 4 equal pressure intervals $\Delta p/2$, such that $\Delta p\equiv p_{k+1}-p_{k-1}$ and $k=0, 1, 2, 3, 4$. We take $\Delta p=500$ mb.

The upper boundary condition must be

$$\omega\equiv 0 \text{ at } k=0. \tag{17}$$

We exclude external gravitational propagation by requiring that $\omega\equiv 0$ at the lower boundary which is taken to coincide with the pressure coordinate surface $p_4=1000$ mb. Therefore, applying (6) at $k=1, 3$, we have that

$$\left.\begin{aligned}&-\mathcal{D}_1=\mathcal{D}_3=\frac{\omega_2}{\Delta p}\\ \text{and}\quad&\\ &\omega_1=\omega_3=\frac{\omega_2}{2}=-\frac{\Delta p}{2}\mathcal{D}_1\end{aligned}\right\} \tag{18}$$

in which ω_1 and ω_3 have been linearly interpolated from the neighboring levels.

Applying the equations of motion (13) and (14) at $k=1, 3$, we have

$$\left.\begin{aligned}\frac{\partial\mathbf{V}_k}{\partial t}&=m\mathbf{G}_k-m\nabla\phi_k\\ \mathbf{G}_k&=-\mathbf{I}_k+\mathbf{F}_k\end{aligned}\right\} \tag{19}$$

where

$$m\mathbf{I}\equiv\mathbf{i}\left[u\frac{\partial u}{\partial x}+mv\frac{\partial u/m}{\partial y}+\omega\frac{\partial u}{\partial p}-\alpha v\left(\frac{u}{a}+2\Omega\right)\right]+\mathbf{j}\left[u\frac{\partial v}{\partial x}+mv\frac{\partial v/m}{\partial y}+\omega\frac{\partial v}{\partial p}+\alpha u\left(\frac{u}{a}+2\Omega\right)\right]. \tag{20}$$

It will be useful at this point to adopt the notation

$$\overline{(\;)}\equiv(\;)_1+(\;)_3;\ \hat{(\;)}\equiv(\;)_1-(\;)_3. \tag{21}$$

The vertical momentum transport in (20) is calculated by evaluating the vertical wind shear non-centrally, and applying the continuity equation (18):

$$\left(\omega\frac{\partial\mathbf{V}}{\partial p}\right)_1=\left(\omega\frac{\partial\mathbf{V}}{\partial p}\right)_3=\frac{\hat{\mathcal{D}}\hat{\mathbf{V}}}{4}. \tag{22}$$

The viscosity will be discussed in Section 2b.

The thermodynamic energy equation (8) is applied at $k=2$. The horizontal wind in (16) is evaluated by the arithmetic mean

$$2\mathbf{V}_2=\mathbf{V}_1+\mathbf{V}_3=\overline{\mathbf{V}}.$$

Hence (8) becomes, upon applying the hydrostatic and gas equations,

$$\frac{\partial\hat{\phi}}{\partial t}=-\left(\frac{\bar{u}}{2}\frac{\partial}{\partial x}+\frac{\bar{v}}{2}\frac{\partial}{\partial y}\right)\hat{\phi}-\gamma^2\hat{\mathcal{D}}+\frac{R}{c_p}Q \tag{23}$$

in which $2\gamma^2\equiv(\Delta p)^2\,(\partial\phi/\partial p)\,(\partial\ln\Theta/\partial p)\approx\hat{\phi}\,\hat{\Theta}/\Theta_2$ is taken as a constant and $Q\equiv\dot{q}\Delta p/p_2$. We shall not specify the nature of the external heat source, Q, since it is not germane to the present discussion.

Upon substituting (22) into (20) then the system (19) and (23) provides 5 scalar equations in 6 unknowns: u_k, v_k, ϕ_k. The sixth equation is provided by the requirement that the mean motions remain nondivergent. Forming $\partial\overline{\mathcal{D}}/\partial t$ from (19) according to (15) and setting it to zero we have

$$\nabla^2\bar{\phi}=\nabla\cdot\overline{\mathbf{G}} \tag{24}$$

which is the "divergence" equation corresponding to the vertically integrated flow. We shall refer to this system as System I.

Alternatively we may define a stream function ψ for the vertically integrated flow:

$$\left.\begin{aligned}\bar{u}&=-m^2\frac{\partial\psi}{\partial y}\\ \bar{v}&=m^2\frac{\partial\psi}{\partial x}.\end{aligned}\right\} \tag{25}$$

Hence the stream function tendency $\psi^*\equiv\partial\psi/\partial t$ in the equations of motion (19) gives:

$$\frac{\partial\bar{u}}{\partial t}=-m^2\frac{\partial\psi^*}{\partial y}=m\overline{G}_x-m^2\frac{\partial\bar{\phi}}{\partial x} \tag{26}$$

$$\frac{\partial\bar{v}}{\partial t}=m^2\frac{\partial\psi^*}{\partial x}=m\overline{G}_y-m^2\frac{\partial\bar{\phi}}{\partial y}. \tag{27}$$

Taking the curl of (26) and (27) yields

$$\nabla^2\psi^*=\text{curl}\,\overline{\mathbf{G}}. \tag{28}$$

This of course is the vorticity equation governing the vertically integrated flow. If we now form the equations of motion for the shear wind $\hat{\mathbf{V}}$ from (19)

$$\frac{\partial\hat{\mathbf{V}}}{\partial t}=m\hat{\mathbf{G}}-m\nabla\hat{\phi} \tag{29}$$

then equations (23), (28), and (29), which will be referred to as System II, constitute 4 scalar equations in the 4 unknowns ψ^*, $\hat{u}$, $\hat{v}$, $\hat{\phi}$. $\hat{\phi}$ is proportional to the specific volume. Because the history of the vertically integrated flow in this system is carried in ψ, $\bar{\phi}$ never is calculated explicitly.

Hence the constraint of filtering the external gravitational solutions, which dictates the elliptic balance condition (24) in System I or (28) in System II, yields a system of equations which is a combined marching-jury problem. Without this constraint, the primitive equations are completely hyperbolic and constitute a pure marching problem.

b. PHYSICAL LATERAL BOUNDARY CONDITIONS AND THE VISCOSITY

We take as the domain of integration a zonal strip bounded by two latitudinal walls at $y=0$ and $y=Y$. The walls are taken to be perfectly smooth. In the x direction we assume cyclic continuity so that all dependent variables and their derivatives are continuous. At the walls we must impose the kinematic boundary condition

$$v_k=0 \text{ at } y=0, Y \text{ for all } t \tag{30}$$

which by (25) and (27) requires the boundaries to be a streamline at each level for all t, giving the corollary physical boundary conditions

$$\psi,\ \psi^* \text{ independent of } x \text{ on } y=0,\ Y. \tag{31}$$

It will suffice for our present purposes to postulate only a lateral viscosity of the Navier-Stokes type. Physically it is desirable that the form of this viscosity be such that the walls do not affect the total zonal angular momentum nor the total energy (through the kinetic energy). This smoothness condition will provide us with a second physical boundary condition. Such a form is

$$\mathbf{F}=Km^3\left\{\mathbf{i}\left[\frac{\partial}{\partial x}\left(\frac{1}{m^2}\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{m^2}\frac{\partial u}{\partial y}\right)\right]+\mathbf{j}\left[\frac{\partial}{\partial x}\left(\frac{1}{m^2}\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{m^2}\frac{\partial v}{\partial y}\right)\right]\right\} \tag{32}$$

where K is assumed constant. We will now demonstrate that this form does indeed possess the above properties.

The change of the total relative zonal angular momentum per unit mass is

$$\frac{\partial}{\partial t}\iint\left(\frac{a^2}{m^2}\dot{\lambda}\right)\frac{a d\lambda}{m}\,a d\theta\equiv\frac{\partial}{\partial t}\int_0^Y\oint au\frac{dxdy}{m^4}. \tag{33}$$

Hence the contribution from $\mathbf{F}$ can be calculated from (19) and (32) to be

$$a\int_0^Y\oint\frac{F_x}{m^3}dxdy=aK\int_0^Y\oint\left[\frac{\partial}{\partial x}\left(\frac{1}{m^2}\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{m^2}\frac{\partial u}{\partial y}\right)\right]dxdy. \tag{34}$$

The first term on the right side must vanish due to the cyclic continuity condition, leaving

$$aK\oint\left[\frac{1}{m^2}\frac{\partial u}{\partial y}\right]_{y=0}^{y=Y}dx. \tag{35}$$

The change of total kinetic energy is

$$\frac{\partial}{\partial t}\iint\frac{a^2}{2}\left[\left(\frac{\dot{\lambda}}{m}\right)^2+(\dot{\theta})^2\right]\frac{a d\lambda}{m}\,a d\theta=\frac{\partial}{\partial t}\int_0^Y\oint\frac{1}{2}(u^2+v^2)\frac{dxdy}{m^4} \tag{36}$$

so that the contribution of $\mathbf{F}$ is

$$\int_0^Y\oint(uF_x+vF_y)\frac{dxdy}{m^3}=-K\int_0^Y\oint\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]\frac{dxdy}{m^2}+K\int_0^Y\oint\left[\frac{\partial}{\partial x}\left(\frac{u}{m^2}\frac{\partial u}{\partial x}+\frac{v}{m^2}\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{u}{m^2}\frac{\partial u}{\partial y}+\frac{v}{m^2}\frac{\partial v}{\partial y}\right)\right]dxdy \tag{37}$$

where we have integrated by parts. The first term on the right side represents the energy dissipation within the atmosphere and is negative definite. The second integral becomes, upon applying the cyclic continuity condition and the kinematic boundary condition (30),

$$K\oint\left[\frac{u}{m^2}\frac{\partial u}{\partial y}\right]_{y=0}^{y=Y}dx. \tag{38}$$

Upon comparing (38) and (35), we observe that to prevent lateral boundary influence on both the total angular momentum and total kinetic energy we must impose the physical boundary condition

$$K\left(\frac{1}{m^2}\frac{\partial u}{\partial y}\right)=0 \text{ on } y=0,\ Y, \tag{39}$$

i.e., the lateral stress must vanish on each boundary individually.

c. INITIAL CONDITIONS AND TIME INTEGRATION

It will be shown in the discussion which follows that a sufficient set of initial conditions are:

$$\hat{u},\ \hat{v},\ \hat{\phi},\ \bar{\zeta}, \text{ given everywhere} \tag{40}$$

where the vertical component of relative vorticity is

$$\zeta=\text{curl}\left(\frac{\mathbf{V}}{m}\right)=\frac{\partial v}{\partial x}-m^2\frac{\partial u/m^2}{\partial y}. \tag{41}$$

From (25), the vertically integrated vorticity may be written as

$$\bar{\zeta}=\nabla^2\psi. \tag{42}$$

By virtue of the two physical boundary conditions (30) and (39) we have the corollary condition

$$\bar{\zeta}=\frac{2\alpha}{a}\bar{u}=-\frac{2\alpha m^2}{a}\frac{\partial\psi}{\partial y} \text{ on } y=0,\ Y. \tag{43}$$

Hence (42) and (43) constitute a Neumann boundary value problem. ψ may thus be determined everywhere to within an arbitrary constant. System (42) and (43) may be transformed into a Dirichlet problem since ψ must be a constant on $y=0,Y$. Taking $\psi=0$ on $y=0$, then integrating (42), we have

$$\left.\begin{aligned}\psi(Y)=&-\frac{Y}{L}\left(\frac{a}{2\alpha}\oint\frac{\bar{\zeta}}{m^2}\,dx\right)_{y=0}\\&+\frac{1}{L}\int_0^Y\left[\int_0^y\left(\oint\frac{\bar{\zeta}}{m^2}\,dx\right)dy\right]dy\\L\equiv&\oint dx.\end{aligned}\right\}\quad(44)$$

$\bar{u}$ and $\bar{v}$ may therefore be calculated on the interior. Note that since ψ must be independent of x on $y=0, Y$, then $\int_0^Y(\bar{u}/m^2)dy$ must be independent of x; i.e., zonally symmetric. Furthermore,cyclic continuity requires that $\oint\bar{v}\,dx=0$. Equation (42) need only be solved initially since the ellipticity condition (24) or alternatively (28) solved at each time insures that $\overline{D}$ remain zero.

With $\bar{u}$ and $\bar{v}$ thus obtained everywhere and $\hat{u}$ and $\hat{v}$ having been prescribed everywhere initially, one can calculate the wind components at each level from the identities (21). The completion of the set of initial data necessary to integrate timewise will depend on whether we employ System I or II.

For System I it is necessary to solve (24), subject to an appropriate boundary condition. This is provided by requiring that (30) be satisfied for all time in (27), resulting in the corollary condition

$$\frac{\partial\bar{\phi}}{\partial y}=\frac{\overline{G}_y}{m}\qquad\text{on } y=0, Y. \tag{45}$$

Hence (24) and (45) constitute a Neumann boundary value problem for which $\overline{G}_y$ and $\nabla\cdot\overline{\mathbf{G}}$ must be known on the boundaries. With $\bar{\phi}$ found as a solution of (24) and $\hat{\phi}$ having been given initially, ϕ_k may be calculated from (21), so that the six dependent variables u_k, v_k, ϕ_k are known initially. $\mathbf{V}_k$ and $\hat{\phi}$ may then be calculated at the next time from (19) and (23). The new $\mathbf{V}_k$ fields are then used to invert (24) giving $\bar{\phi}$ and hence ϕ_k at the new time. Thus all of the initial dependent variables have been reconstructed. System I is in essence the one proposed by Eliassen [4].

To proceed by means of System II, we need first to determine the corollary boundary conditions for (28). These are obtained by integrating (26) over the entire region and then applying the cyclic continuity condition:

$$\psi^*(Y)-\psi^*(0)=-\frac{1}{L}\int_0^Y\oint\frac{\overline{G}_x}{m}\,dxdy. \tag{46}$$

Since ψ^* is independent of x on $y=0$, Y, we set the arbitrary datum

$$\psi^*(0)=0 \tag{47}$$

so that

$$\psi^*(Y)=-\frac{1}{L}\int_0^Y\oint\frac{\overline{G}_x}{m}\,dxdy. \tag{48}$$

Since the initial ψ has been set to zero on $y=0$, then ψ must remain zero on $y=0$ for all time.

Therefore, System II requires the solution of a Dirichlet boundary value problem. The field ψ^* together with the initial ψ then permits us to calculate $\overline{\mathbf{V}}$ at the next time. Equation (29), which depends on $\hat{\phi}$ but not on $\bar{\phi}$, gives us the new $\hat{\mathbf{V}}$. As before $\overline{\mathbf{V}}$ and $\hat{\mathbf{V}}$ together yield $\mathbf{V}_k$. Finally (23) gives the new $\hat{\phi}$. The data have therefore been reconstructed.

It will be useful at this point to digress for the purpose of discussing some of the computational stability characteristics of the system of equations with which we are dealing.

3. COMPUTATIONAL STABILITY

The Courant-Friedrichs-Lewy (CFL) stability criterion is for the most part governed by the speed of the internal gravitational waves relative to the mesh (Eliassen [4]):

$$\left(\frac{|\mathbf{V}|_{\max}}{2}+\gamma\right)\sqrt{2}\leq\frac{\Delta s}{\Delta t} \tag{49}$$

where Δs is the horizontal grid distance on the earth, and Δt the time increment. Suppose we take $\Delta\lambda$ to be 5° longitude, then at the equator $\Delta s=555$ km. If we take the zonal channel to be 17 grid lengths wide, with $y=0$ at the equator, then $y=Y$ corresponds to 64.4° latitude. Therefore Δs has its smallest value, 240 km., at the north boundary. For an average value of the static stability, $\gamma=60$ m. sec.$^{-1}$. Then for $\Delta t=20$ min. the criterion is fulfilled when

$$\frac{|\mathbf{V}|_{\max}}{2}\leq 80\text{m. sec.}^{-1}$$

Hence one would presume that if (49) were satisfied, the numerical integration should remain stable under the customary techniques of centered differences. The experience of a number of research workers in the past 6 or 7 years has been that attempts at numerical integration of very simple physical systems (such as non-viscous barotropic flows) in the framework of the primitive equations, have resulted in spurious inertio-gravitational oscillations which obscured the meteorologically significant motions even when the CFL criterion was satisfied. Investigations by the Princeton group (Charney [1]) disclose that one cause can be an incorrectly specified initial velocity field. In the present case this corresponds to $\overline{\mathbf{V}}$ which is never specified independently but is derived from the initially specified vertically integrated vorticity, $\bar{\zeta}$, through equation (42). Another possibility offered by Charney for the apparently spurious oscillations is computational instability due to incorrect boundary conditions. It is this latter source of instability that will be dwelt upon here. The nature of the instability will be demonstrated in less rigorous fashion than is normally possible by an analysis of the amplification of small-scale motions.

As a matter of convenience we shall refer to instability resulting from incorrect boundary conditions as *computational instability of the second kind* (in contrast to the CFL instability which may be considered as that of the *first kind*). As it turns out this instability is already possible in the linear zonally symmetric equations corresponding to the model described in the previous section. This is rather fortunate since such a simple system lends itself to a rather clear-cut analysis.

Let us consider zonally symmetric perturbations on a vertically integrated flow which is at rest. At first we will deal with non-viscous, thermally inactive motions.

Assuming for the present that α and m may be replaced by their average values, then equation (23) becomes

$$\frac{\partial\hat{\phi}}{\partial t}=-\gamma^2\frac{\partial\hat{v}}{\partial y} \tag{50}$$

and the equations of motion for the shear flow (29) become

$$\frac{\partial\hat{u}}{\partial t}=2\alpha\Omega\hat{v} \tag{51}$$

$$\frac{\partial\hat{v}}{\partial t}=-2\alpha\Omega\hat{u}-m^2\frac{\partial\hat{\phi}}{\partial y}. \tag{52}$$

We take the domain as before to lie between two latitudinal walls so that the physical boundary condition is

$$\hat{v}=0 \text{ on } y=0, Y \text{ for all } t. \tag{53}$$

The initial conditions are $\hat{\phi}$, $\hat{u}$, $\hat{v}$, given everywhere, the latter subject to (53). Also, to satisfy (52) and (53), $2\alpha\Omega\hat{u}+m^2\partial\hat{\phi}/\partial y\equiv 0$ on the boundaries initially as well as in the interior of time. Hence the time-dependent equations (50–52) constitute a complete set and we have a pure marching problem.

We form the difference analogues of the three first-order equations (50–52), employing central differences over intervals Δt and Δy where $t=\tau\Delta t$ and $y=j\Delta y$, $0\leq j\leq (J-1)$.

$$\frac{\hat{\phi}_j^{\tau+1}-\hat{\phi}_j^{\tau-1}}{2\Delta t}=-\frac{\gamma^2}{2\Delta y}(\hat{v}_{j+1}^{\tau}-\hat{v}_{j-1}^{\tau}) \tag{54}$$

$$\frac{\hat{u}_j^{\tau+1}-\hat{u}_j^{\tau-1}}{2\Delta t}=f\hat{v}_j^{\tau} \tag{55}$$

$$\frac{\hat{v}_j^{\tau+1}-\hat{v}_j^{\tau-1}}{2\Delta t}=-f\hat{u}_j^{\tau}-\frac{m^2}{2\Delta y}(\hat{\phi}_{j+1}^{\tau}-\hat{\phi}_{j-1}^{\tau}). \tag{56}$$

It will be instructive to form a single differential equation in $\hat{v}$ from (50–52)

$$\frac{\partial^2\hat{v}}{\partial t^2}=-f^2\hat{v}+\Gamma^2\frac{\partial^2\hat{v}}{\partial y^2} \tag{57}$$

where $f\equiv 2\alpha\Omega$, $\Gamma\equiv m\gamma$. Then differencing (57) centrally, we have

$$\frac{\hat{v}_j^{\tau+1}-2\hat{v}_j^{\tau}+\hat{v}_j^{\tau-1}}{(\Delta t)^2}=-f^2\hat{v}_j^{\tau}+\frac{\Gamma^2}{(\Delta y)^2}(\hat{v}_{j+1}^{\tau}-2\hat{v}_j^{\tau}+\hat{v}_{j-1}^{\tau}). \tag{58}$$

TABLE 1.—*Compatibility of three dependent variables $\hat{v}$, $\hat{u}$, and $\hat{\phi}$.*

		Is physical boundary condition satisfied at—	
		$j=0$	$j=(J-1)$
Even J	Solutions for $\hat{v}$, $\hat{u}$ at odd j and $\hat{\phi}$ at even j	No	Yes
	Solutions for $\hat{v}$, $\hat{u}$ at even j and $\hat{\phi}$ at odd j	Yes	No
Odd J	Solutions for $\hat{v}$, $\hat{u}$ at odd j and $\hat{\phi}$ at even j	No	No
	Solutions for $\hat{v}$, $\hat{u}$ at even j and $\hat{\phi}$ at odd j	Yes	Yes

If we use (56) initially to obtain $\hat{v}_j^1$ as a function of $\hat{v}_j^{-1}$ in terms of the initial conditions, $\hat{\phi}_j^0$, $\hat{u}_j^0$, $\hat{v}_j^0$, given everywhere, then (58) may be solved as a marching problem with no difficulty in satisfying (53) at both boundaries $j=0$, $(J-1)$. This process will proceed stably provided the CFL condition is met. However, if we form the difference equation in $\hat{v}$ from the first-order difference equations (54–56)

$$\frac{\hat{v}_j^{\tau+2}-2\hat{v}_j^{\tau}+\hat{v}_j^{\tau-2}}{(2\Delta t)^2}=-f^2\hat{v}_j^{\tau}+\frac{\Gamma^2}{(2\Delta y)^2}(\hat{v}_{j+2}^{\tau}-2\hat{v}_j^{\tau}+\hat{v}_{j-2}^{\tau}) \tag{59}$$

we find it exactly in the form of (58) except that (59) applies to double time and space intervals. Therefore, the ratio $\Delta t/\Delta y$ is preserved and the CFL criterion remains the same. We now note that in (59) $\hat{v}$ is linked only at alternate values of j as well as of τ.* Consider the case when J is odd. Then (59) applied at even j satisfies (53) at both boundaries. On the other hand, application of (59) at odd j cannot directly satisfy (53) at either boundary since the finite difference equivalent of $\partial\hat{v}/\partial y$ is required. Alternatively, consider the case of even J. Now the application of (59) at even j will satisfy the physical boundary condition at $j=0$ but not at $j=(J-1)$, whereas solutions at odd j will not satisfy (53) at $j=0$ but will at $j=(J-1)$. Therefore, neither solution is compatible with the physical conditions at both boundaries. That is, solutions at even j require $\partial\hat{v}/\partial y$ at $j=(J-1)$, and solutions at odd j require $\partial\hat{v}/\partial y$ at $j=0$. Returning to the system of first-order equations (54–56), we can also see the consequences on $\hat{u}$ and $\hat{\phi}$ for even and odd J. The compatibility of the three dependent variables is summarized in table 1.

It is clear that corollary boundary conditions can be deduced from the system of differential equations (50–52) and the physical condition (53). From (51) we see that $\partial\hat{u}/\partial t=0$; differentiating (52) timewise yields $\partial^2\hat{\phi}/\partial t\partial y=0$; differentiating (50) with respect to y yields $\partial^2\hat{v}/\partial y^2=0$. These, however, do not provide the conditions for $\partial\hat{v}/\partial y$.

The condition on $\partial\hat{v}/\partial y$ must be such as to yield compatible solutions at adjacent points and hence must depend not only on the differential equations and the physical boundary conditions, but also on their form when differencing is performed and on the method of differencing. We will refer to such conditions as *computational boundary conditions*.

*During the preparation of this manuscript the writer's attention was drawn to a paper by Platzman [7] which also points out this property of central differencing techniques.

Heuristically it appears reasonable that numerical integration between the boundaries of the difference analogue of the quantity for which a computational boundary condition is required must correspond exactly to the integral of its continuous form.

Let this quantity be denoted in general by $\partial\chi/\partial y$, where χ is known on $y=0,Y$ as a physical condition or as a corollary. Therefore the integral of $\partial\chi/\partial y$:

$$\int_0^Y \frac{\partial\chi}{\partial y}dy=\chi(Y)-\chi(0) \tag{60}$$

is exact and known. The finite difference sum over *all* points equivalent to the left side of (60) is

$$\left[\frac{1}{2}\left(\frac{\partial\chi}{\partial y}\right)_0+\sum_{j=1}^{J-2}\left(\frac{\partial\chi}{\partial y}\right)_j+\frac{1}{2}\left(\frac{\partial\chi}{\partial y}\right)_{J-1}\right]\Delta y. \tag{61}$$

In the case of central differences

$$\left(\frac{\partial\chi}{\partial y}\right)_j=\frac{\chi_{j+1}-\chi_{j-1}}{2\Delta y}. \tag{62}$$

Substituting (62) in (61) and equating to the right side of (60)

$$\frac{1}{4}\left[(\chi_1-\chi_{-1})+(\chi_J-\chi_{J-2})\right]+\frac{1}{2}\sum_{j=1}^{J-2}(\chi_{j+1}-\chi_{j-1})=\chi_{J-1}-\chi_0. \tag{63}$$

Upon carrying out the indicated summation we have

$$(\chi_J-2\chi_{J-1}+\chi_{J-2})=(\chi_1-2\chi_0+\chi_{-1}) \tag{64}$$

which is the difference analogue of the condition that $\partial^2\chi/\partial y^2$ be equal at the boundaries. A *sufficient* condition to satisfy (64) is that the left and right sides vanish individually so that

$$\left.\begin{aligned}\left(\frac{\partial\chi}{\partial y}\right)_0&=\frac{1}{2\Delta y}(\chi_1-\chi_{-1})=\frac{1}{\Delta y}(\chi_1-\chi_0)\\ \left(\frac{\partial\chi}{\partial y}\right)_{J-1}&=\frac{1}{2\Delta y}(\chi_J-\chi_{J-2})=\frac{1}{\Delta y}(\chi_{J-1}-\chi_{J-2}).\end{aligned}\right\} \tag{65}$$

Equations (65) are the required computational boundary conditions. They are equivalent to the requirement that $\partial\chi/\partial y$ be calculated at the boundaries by means of one-sided differences over a single grid interval. This result is intuitively acceptable and might have been arrived at without the *a priori* requirement that the exact integral condition be satisfied. It is of interest that Phillips, in a recent successful integration of the barotropic primitive equations for a fluid with a free surface in a hemispheric domain bounded by an equatorial wall, applied anti-symmetry conditions on the wind component normal to the boundary.* This may be deduced as a consequence of (64). The exact integral condition has provided a sufficient condition for deriving the computational boundary conditions. The sufficiency has only been established empirically; i. e. through extended period integrations. It is not as yet clear what the *necessary and sufficient* conditions must be.

*These results are as yet unpublished, but a reference to this condition is given in an earlier paper by Phillips [6].

It is to be emphasized that the computational boundary conditions will depend on the form of the differential equations which are differenced and the difference technique; for instance, whether derivations of products are carried out before differencing. In particular (58) does not require any computational boundary conditions at all. The case of system (54–56), as we have seen, requires $\partial\hat{v}/\partial y$. Applying the physical condition (53) to (65) we have

$$\left.\begin{aligned}\left(\frac{\partial\hat{v}}{\partial y}\right)_0&=\frac{\hat{v}_1}{\Delta y}\\ \left(\frac{\partial\hat{v}}{\partial y}\right)_{J-1}&=-\frac{\hat{v}_{J-2}}{\Delta y}\end{aligned}\right\} \tag{66}$$

and the problem for zonally-symmetric linear motion is completely stated for numerical integration.

Let us now proceed to a somewhat more complex case—that of viscous flow with external heating RQ/c_p, which will assume a given function of y.

We will now consider the effect of a lateral viscosity and heating in the linear zonally symmetric system. The system of equations is then

$$\frac{\partial\hat{\phi}}{\partial t}=-\gamma^2\frac{\partial\hat{v}}{\partial y}+\frac{R}{c_p}Q \tag{67}$$

$$\frac{\partial\hat{u}}{\partial t}=f\hat{v}+m^2K\frac{\partial^2\hat{u}}{\partial y^2} \tag{68}$$

$$\frac{\partial\hat{v}}{\partial t}=-f\hat{u}-m^2\frac{\partial\hat{\phi}}{\partial y}+m^2K\frac{\partial^2\hat{v}}{\partial y^2}. \tag{69}$$

From the considerations of the non-symmetric system in Section 2, we have two physical boundary conditions

$$\hat{v}=0 \text{ on } y=0,\ Y \tag{70}$$

$$\frac{\partial\hat{u}}{\partial y}=0 \text{ on } y=0,\ Y. \tag{71}$$

The initial conditions are the same as before: $\hat{\phi}$, $\hat{u}$, $\hat{v}$ given everywhere subject to (70), and we still have a pure marching problem. On the boundaries $\partial\hat{v}/\partial t$ is known from (70), and $\partial\hat{\phi}/\partial t$ in (67) is known if $\partial\hat{v}/\partial y$ is calculated from the computational boundary condition (66), as before. For $\partial\hat{u}/\partial t$ in (68) we need an additional computational condition on $\partial^2\hat{u}/\partial y^2$. This is obtained by taking $\partial\hat{u}/\partial y$ for χ in (65) and applying (71), then the exact integral condition yields:

$$\left.\begin{aligned}\left(\frac{\partial^2\hat{u}}{\partial y^2}\right)_0&=\frac{1}{\Delta y}\left(\frac{\partial\hat{u}}{\partial y}\right)_1=\frac{1}{2(\Delta y)^2}(\hat{u}_2-\hat{u}_0)\\ \left(\frac{\partial^2\hat{u}}{\partial y^2}\right)_{J-1}&=-\frac{1}{\Delta y}\left(\frac{\partial\hat{u}}{\partial y}\right)_{J-2}=-\frac{1}{2(\Delta y)^2}(\hat{u}_{J-1}-\hat{u}_{J-3}).\end{aligned}\right\} \tag{72}$$

It should be pointed out that central time and space differences applied to (68) and (69) will, due to the viscosity, give rise to equations of the form

$$\hat{u}_j^{\tau+1}-\hat{u}_j^{\tau-1}= \ldots -\left(\frac{m^2K4\Delta t}{(\Delta y)^2}\right)\hat{u}_j^{\tau}. \tag{73}$$

Hence a spurious computational solution is introduced that is unstable when the coefficient $K>0$, which it is in this case. One means for avoiding it is to evaluate $\hat{u}_j$ on the right side at $(\tau-1)$. We may refer to this as *computational instability of the third kind,* which is thoroughly discussed by Eliassen [3] and Richtmyer [8].

4. NON-LINEAR BAROCLINIC FLOWS (CONTINUED)

a. COMPUTATIONAL BOUNDARY CONDITIONS

We may now proceed to complete the discussion of the numerical integration of the fully non-linear, zonally asymmetric system described in Section 2.

It is appropriate at this time to compare the merits of Systems I and II. Perhaps the most important consideration is their relative stability under numerical integration. The purpose of the elliptic equations is to insure that $\partial\overline{D}/\partial t\equiv 0$. System II does this directly by regenerating the stream function. Hence by definition, truncation and round-off errors cannot introduce divergence into the $\overline{\mathbf{V}}$ field computed from it. On the other hand, by working through $\overline{\phi}$ in System I, there is no way of avoiding degeneracy (i.e., the introduction of spurious $\overline{D}\neq 0$) due to truncation and round-off except through periodic rebalancing by means of (42), assuming $\overline{\zeta}$ to be essentially correct. One further advantage in favor of System II is that the numerical solution of Dirichlet problems by relaxation methods seems to converge more rapidly than that for Neumann problems. This may be due only to the fact that we have far greater experience with Dirichlet problems. Nevertheless, it is an important economical consideration where extended-period calculations are contemplated. Therefore, we will confine ourselves to consideration of System II.

It is desirable that the form of the continuous equations to be differenced be such that time changes of zonal angular momentum and temperature possess exact integrals over the entire area. That is, we wish to avoid spurious sources of angular momentum and heating due to truncation error in the non-linear terms. It is clear that the potential and kinetic energy integrals will not be exact. Furthermore one should avoid terms of the form (73) from appearing in non-viscous terms if there is a possibility of computational instability of the third kind. Thus applying (18) to (23), and (18) and (22) to (20), we have

$$\frac{\partial\hat{\phi}}{dt}=-\frac{\partial}{\partial x}\left(\frac{\hat{\phi}\overline{u}}{2}\right)-m^2\frac{\partial}{\partial y}\left(\frac{\hat{\phi}\overline{v}}{2m^2}\right)-\gamma^2\hat{D}+\frac{R}{c_p}Q \tag{74}$$

$$\left.\begin{aligned}
\mathsf{I}_1=\mathbf{i}&\left[\frac{\partial u_1^2}{\partial x}+m^4\frac{\partial}{\partial y}\left(\frac{u_1v_1}{m^4}\right)-\frac{\hat{D}\overline{u}}{4}-2\Omega\alpha v_1\right]\\
+\mathbf{j}&\left[\frac{\partial u_1v_1}{\partial x}+m^3\frac{\partial}{\partial y}\left(\frac{v_1^2}{m^3}\right)-\frac{\hat{D}\overline{v}}{4}+\alpha\left(2\Omega+\frac{u_1}{a}\right)u_1\right]\\
\mathsf{I}_3=\mathbf{i}&\left[\frac{\partial u_3^2}{\partial x}+m^4\frac{\partial}{\partial y}\left(\frac{u_3v_3}{m^4}\right)+\frac{\hat{D}\overline{u}}{4}-2\Omega\alpha v_3\right]\\
+\mathbf{j}&\left[\frac{\partial u_3v_3}{\partial x}+m^3\frac{\partial}{\partial y}\left(\frac{v_3^2}{m^3}\right)+\frac{\hat{D}\overline{v}}{4}+\alpha\left(2\Omega+\frac{u_3}{a}\right)u_3\right].
\end{aligned}\right\} \tag{75}$$

It is clear that because of the cyclic continuity condition in x, computational boundary conditions may be necessary only on the zonal boundaries. We have seen in Section 2c that the boundary value problem to construct the initial ψ field everywhere is completely stated for numerical integration, without need for a computational boundary condition. This is also true for the initial $\overline{\mathbf{V}}$ field.

In the time integration of the geopotential thickness, $\hat{\phi}$, in (74), we need $\hat{D}$ on the boundaries, and also $m^2\partial(\hat{\phi}\overline{v}/2m^2)/\partial y$. Note that the latter would not have been necessary had we used the form in (23) since $\overline{v}$ vanishes on the boundary. Applying (65) we have

$$\left.\begin{aligned}
(\hat{D})_{i,0}&=\left(\frac{\partial\hat{u}}{\partial x}\right)_{i,0}+\frac{m_0^2}{m_1^2\Delta}(\hat{v})_{i,1}\\
(\hat{D})_{i,J-1}&=\left(\frac{\partial\hat{u}}{\partial x}\right)_{i,J-1}-\frac{m_{J-1}^2}{m_{J-2}^2\Delta}(\hat{v})_{i,J-2}
\end{aligned}\right\} \tag{76}$$

and

$$\left.\begin{aligned}
\left[m^2\frac{\partial}{\partial y}\left(\frac{\hat{\phi}\overline{v}}{2m^2}\right)\right]_{i,0}&=\frac{m_0^2}{2m_1^2\Delta}(\hat{\phi}\overline{v})_{i,1}\\
\left[m^2\frac{\partial}{\partial y}\left(\frac{\hat{\phi}\overline{v}}{2m^2}\right)\right]_{i,J-1}&=-\frac{m_{J-1}^2}{2m_{J-2}^2\Delta}(\hat{\phi}\overline{v})_{1,J-2}
\end{aligned}\right\} \tag{77}$$

where we have taken $\Delta\equiv\Delta x=\Delta y$.

To calculate inertial terms, I, on the boundary from (75) we need in addition:

$$\left.\begin{aligned}
\left[m^4\frac{\partial}{\partial y}\left(\frac{uv}{m^4}\right)\right]_{i,0}&=\frac{m_0^4}{m_1^4\Delta}(uv)_{i,1}\\
\left[m^4\frac{\partial}{\partial y}\left(\frac{uv}{m^4}\right)\right]_{i,J-1}&=-\frac{m^4{}_{J-1}}{m^4{}_{J-2}\Delta}(uv)_{i,J-2}
\end{aligned}\right\} \tag{78}$$

$$\left.\begin{aligned}
\left[m^3\frac{\partial}{\partial y}\left(\frac{v^2}{m^3}\right)\right]_{i,0}&=\frac{m_0^3}{m_1^3\Delta}(v^2)_{i,1}\\
\left[m^3\frac{\partial}{\partial y}\left(\frac{v^2}{m^3}\right)\right]_{i,J-1}&=-\frac{m^3{}_{J-1}}{m^3{}_{J-2}\Delta}(v^2)_{i,J-2}
\end{aligned}\right\} \tag{79}$$

The frictional force (32) requires

$$\left.\begin{aligned}
\left[m^3\frac{\partial}{\partial y}\left(\frac{1}{m^2}\frac{\partial u}{\partial y}\right)\right]_{i,0}&=\frac{m_0^3}{m_1^2 2\Delta^2}(u_{i,2}-u_{i,0})\\
\left[m^3\frac{\partial}{\partial y}\left(\frac{1}{m^2}\frac{\partial u}{\partial y}\right)\right]_{i,J-1}&=-\frac{m_{J-1}^3}{m_{J-2}^2 2\Delta^2}(u_{i,J-1}-u_{i,J-3})
\end{aligned}\right\} \tag{80}$$

and

$$\left.\begin{aligned}\left[m^3\frac{\partial}{\partial y}\left(\frac{1}{m^2}\frac{\partial v}{\partial y}\right)\right]_{i,0}&=\frac{m_0^3}{\Delta}\left[\frac{v_{i,2}}{2m_1^2\Delta}-\frac{(\partial v/\partial y)_{i,0}}{m_0^2}\right]\\ \left[m^3\frac{\partial}{\partial y}\left(\frac{1}{m^2}\frac{\partial v}{\partial y}\right)\right]_{i,J-1}&=\frac{m_{J-1}^3}{\Delta}\left[\frac{(\partial v/\partial y)_{i,J-1}}{m_{J-1}^2}+\frac{v_{i,J-3}}{2m_{J-2}^2\Delta}\right].\end{aligned}\right\} \tag{81}$$

In (81), $\partial v/\partial y$ on the boundary must be evaluated consistently with the calculation of the divergence on the boundary. For example, at $j=0$ and referring to (65) and (76) we have that

$$\left(\frac{v}{m^2}\right)_{i,-1}=-\left(\frac{v}{m^2}\right)_{i,1};$$

therefore

$$\left(\frac{\partial v}{\partial y}\right)_{i,0}=\frac{1}{2\Delta}\left(1+\frac{m_{-1}^2}{m_1^2}\right)v_{i,0}. \tag{82}$$

Similarly on the other boundary

$$\left(\frac{\partial v}{\partial y}\right)_{i,J-1}=-\frac{1}{2\Delta}\left(1+\frac{m_J^2}{m_{J-2}^2}\right)v_{i,J-2}. \tag{83}$$

Hence $\mathbf{G}_k$ is known everywhere and the stream function tendency may be calculated from (28) and (48).

b. COMPUTATIONAL ASPECTS OF THE ELLIPTIC PART

We may use the customary extrapolated Liebmann relaxation technique to calculate the initial ψ from (42) and ψ^* from (28). The first guess for ψ may be obtained by integrating (42) with boundary condition (43)

$${}^0\psi(y)=\frac{1}{L}\oint\psi dx=-\frac{y}{L}\left(\frac{a}{2\alpha}\oint\frac{\bar{\zeta}}{m^2}dx\right)_{y=0}+\frac{1}{L}\int_0^y\left[\int_0^y\left(\oint\frac{\bar{\zeta}}{m^2}dx\right)dy\right]dy. \tag{84}$$

The anterior superscript denotes the iterative index ν.

Equation (84) may also be employed to hasten convergence. As each row is relaxed within a given scan, ν, the mean value of ${}^\nu\psi_{i,j}$ over all i must satisfy (84). The ψ's are then adjusted accordingly before going on to the next row.

In the case of ψ^* we have the source of a better guess through extrapolation:

$$\left.\begin{aligned}\tau=0&: {}^0\psi^{*0}=0\\ \tau=1&: {}^0\psi^{*1}=\psi^{*0}\\ \tau>1&: {}^0\psi^{*\tau}=2\psi^{*\tau-1}-\psi^{*\tau-2}.\end{aligned}\right\} \tag{85}$$

We may accelerate convergence of the relaxation of (28) by again adjusting the mean value of a newly relaxed row to the integral of (26):

$$\oint\psi^*dx=-\int_0^y\left(\oint\frac{\overline{G}_x}{m}dx\right)dy. \tag{86}$$

This technique should be applied for only a few scans since the error due to adjustment quickly becomes comparable to the iterative error during the process of convergence. Such integrations have been performed on a grid of 18 x 72 points with a relaxation factor of 1.25 and a criterion $|{}^{\nu+1}\psi^*-{}^\nu\psi^*|/g<(15/64)m$, where $g=9.81$ m. sec^{-2}. The application of (86) was stopped when $|{}^{\nu+1}\psi^*-{}^\nu\psi^*|/g<(75/64)$m. The number of iterations necessary for convergence varied between 2 and 6.

c. CONSTRUCTION OF THE ϕ FIELD

Although the $\bar{\phi}$ and ϕ_k fields never enter explicitly into System II, it may be of interest to examine these quantities during the course of the calculation. This can be accomplished through integration of equations (26–27) by simple quadratures. In doing this numerically, care must be taken to avoid accumulation of systematic truncation error. That is, the numerical solution of equations (26–27) should be independent of the path taken for the numerical quadratures. Consider the scheme

$$\bar{\phi}_{i+\frac{1}{2},j+\frac{1}{2}}-\bar{\phi}_{i-\frac{1}{2},j+\frac{1}{2}}=(\psi^*_{i,j+1}-\psi^*_{i,j})+\frac{\Delta}{2}\left[\left(\frac{\overline{G}_x}{m}\right)_{i,j+1}+\left(\frac{\overline{G}_x}{m}\right)_{i,j}\right] \tag{87}$$

$$\bar{\phi}_{i+\frac{1}{2},j+\frac{1}{2}}-\bar{\phi}_{i+\frac{1}{2},j-\frac{1}{2}}=-(\psi^*_{i+1,j}-\psi^*_{i,j})+\frac{\Delta}{2}\left[\left(\frac{\overline{G}_y}{m}\right)_{i+1,j}+\left(\frac{\overline{G}_y}{m}\right)_{i,j}\right]. \tag{88}$$

Upon eliminating $\bar{\phi}$ between (87) and (88), we have

$$\psi_{i,j+1}+\psi^*_{i,j-1}+\psi^*_{i+1,j}+\psi^*_{i-1,j}-4\psi^*_{i,j}=\frac{\Delta}{2}\left[-\left(\frac{\overline{G}_x}{m}\right)_{i,j+1}+\left(\frac{\overline{G}_x}{m}\right)_{i,j-1}+\left(\frac{\overline{G}_y}{m}\right)_{i+1,j}-\left(\frac{\overline{G}_y}{m}\right)_{i-1,j}\right]$$

which is precisely the difference analogue of (28). Setting an arbitrary datum at a point, say $\bar{\phi}_{\frac{1}{2},-\frac{1}{2}}=0$, then one can calculate $\bar{\phi}_{\frac{1}{2},j+\frac{1}{2}}$ for $j=0,\ldots,(J-1)$ from (88). Since $i=i+I$, then with (87) we can obtain $\bar{\phi}_{i+1\frac{1}{2},j+\frac{1}{2}}$ for $i=0,\ldots,(I-2)$; $j=0,\ldots,(J-2)$. Finally, we employ (88) again to calculate $\bar{\phi}_{i+1\frac{1}{2},-\frac{1}{2}}$ and $\bar{\phi}_{i+1\frac{1}{2},J-\frac{1}{2}}$ for $i=0,\ldots,(I-2)$. We now have $\bar{\phi}_{i+\frac{1}{2},j+\frac{1}{2}}$ for $i=0,\ldots,(I-1)$; $j=-1,0,1,\ldots,(J-1)$. This is sufficient for an interpolation of $\bar{\phi}$ at the boundaries with a correct representation of $\partial\bar{\phi}/\partial y$ at the boundaries.

5. CONCLUDING REMARKS

We have developed a scheme for numerically integrating the baroclinic primitive equations over a domain with closed boundaries. In actual application, the methods described have yielded an analogue of the primitive equations which is stable when integrated numerically over extended periods. At a time when it appeared unclear how numerical integration of the primitive equations could be made free of spurious gravitational instability, J. von Neumann proposed the inclusion of a compressional viscosity in the equations of motion to suppress the amplitude of gravitational disturbances of wavelength comparable to grid size. Since such a viscosity has no physical counterpart, one would expect a systematic distortion of the evolving motions. However, it appears (empirically) that the method discussed here for formulating the computational boundary conditions precludes the occurrence of spurious gravitational instability. Hence the use of an artificial compressional viscosity may be removed from consideration.

The exact integral condition for deducing computational boundary conditions must apply as well to the form of the baroclinic primitive equations which also admit external gravitational motions. Those meteorological studies for domains where flow through the boundaries is permitted present a special problem. The reason of course is that a correct statement of the appropriate physical boundary conditions is not clear. The investigation of Platzman [7] is a significant contribution in this direction.

Experience has demonstrated that a consistent use of the geostrophic approximation can yield great insight into the large-scale atmospheric processes in spite of the obvious limitations wrought by the restrictive approximations. Historically, linear techniques have played a similar role in providing a fundamental understanding of dynamical processes at the expense of relatively little mathematical effort. It appears that the step to completeness from linear and geostrophic investigations is most profitably made by going directly to the nonlinear primitive equations. The fundamental simplicity and self consistency inherent in the primitive equations, together with an assurance of stability under numerical treatment, would seem to suggest this as the logical course. The use of the balance equations as an intermediate step offers questionable diagnostic gain. This conclusion is based on the dubious increase of understanding gained in return for considerable computational complexity.

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