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Some aspects of the general circulation

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I would like to express my deep gratitude to the Royal Meteorological Society for this great and unique opportunity to address you on this occasion.

My first thought was to present an exhaustive survey and review of what we now know about the atmosphere's general circulation, but I reconsidered and decided instead to reflect on where we are going, what we hope to learn, and what some of the more obvious obstacles are. Since one of the most promising tools for research in the general circulation is the high-speed computer – its role and its pitfalls will deserve some attention.

As a fluid system, the earth's atmosphere could presumably admit a tremendous variety of motions – considering the many sources of excitation. Perhaps, viewing the atmosphere locally, on a small scale, this would appear to be a valid conclusion. However, the large-scale weather-producing motions, those responsible for our climate, are far from random. In fact, the atmosphere in-the-large possesses a remarkably stable climatology. And yet it is the small deviations from these climatic properties that we are most interested in – the determination of which we assign to the weather forecaster. To do this in a scientifically deterministic way rather than statistically, requires that we first understand the selective processes that give the tremendous stability to our atmosphere's climatology – that is, we must develop a theory of climatogenesis – to coin a word.

As a prelude, I would like to emphasize some of the striking differences between the short term ($\sim$ 1 day) and the long term (say $\sim$ 1) month) evolutions of the atmosphere.

The *short-period evolutions* are hardly influenced by the slow acting external forces. The motions are inertial, behaving essentially as free disturbances tending to establish equilibrium. Hence, the details of the initial state are important since they primarily determine the nature and extent of the imbalance. As a consequence the climatological features of the general circulation are already inherent in the initial data. A dynamical model must therefore correctly interpret the initial imbalance in order to predict the short-term inertial evolution of the free disturbances, and at the same time not destroy their meteorological or climatological character.

On the other hand, the *longer-period evolutions* are essentially controlled by the energy sources and sinks. They do not really pose an initial value problem in the same sense since the atmosphere, a dissipative system, tends to forget the details of its state in the distant past. Hence only the statistical properties rather than the details of the inertial dynamics should be important. Therefore, in studying the general circulation we must ask why there are stable properties of the atmosphere which vary over only a relatively small range, e.g., the characteristic wind distribution; the thermal structure : i.e. the lapse rate; the existence of the tropopause; the quasi-geostrophic and quasi-barotropic character; the characteristic time and space scales of disturbances. These properties in turn must be related to the composition of the atmosphere, the kinematical and thermal properties of the earth's surface, the size of the earth, its rotation rate, and the solar radiation.

The short-term evolutions may therefore be viewed as a special case of the broader problem of the long-term evolutions. Whereas the former can mainly be dealt with in the first approximation in terms of the dynamics of a free barotropic fluid, the latter, in its fullest complexity, encompasses all which we call meteorology and perhaps more : not only transformations between potential and kinetic energy but also the interplay of radiative transfer, of the dynamically determined distribution of radiatively active gases, the hydrologic cycle including cloud physics, convective and lateral small-scale turbulent exchanges, boundary layer interactions, etc. In fact, it is doubtful whether we may ultimately treat the atmosphere and oceans as independent systems.

Which brings me to the question of what we mean by ' the general circulation.' At one time or another it has been all things to all men. But most common perhaps it connotes the climatology of *atmospheric motions*. Too often this is equated to a static collection of statistics and a geometric description. It has become apparent that an adequate description of the whole of climatology, requires higher statistical moments, not only of the motion field, but also of the other physically significant dependent variables and their interactions. Such a description at once becomes a meaningful dynamical tool – diagnostic in character and coherent in implication. The higher moments have precise interpretations in terms of the fluxes of angular momentum, of heat, of moisture, of kinetic energy, and of transformations of energy between various types and various scales. The climatology and the climatogenesis are then virtually indistinguishable and inextricable. Furthermore, such a description has the advantage of being succinct. I need only mention that in the conventional geometric description of atmospheric structure, it takes of the order of a million dependent variables to define the large-scale state of the lower 99 per cent of the atmosphere at any one time.

Although we are forever trying to construct more complex models of the atmosphere – the most significant advances in meteorology have come from studies which have isolated essential mechanisms. Meteorology is after all an applied science, in the sense that we apply the fundamental results of classical physics : Newton's equations of motion, the law of conservation of mass, the first law of thermodynamics, the equation of state. Of the numerous processes accounted for by these laws, we usually ask : What subset are mainly responsible for what we observe ? The burden of the theoretical meteorologist has been to make an educated guess in order to reduce the complexity of the fundamental laws to a mathematically tractable form and then to compare the results with observation. In this way many significant advances have been made : the quasi-static theory, the applicability of the theory of vorticity conservation, the theory for the instability of baroclinic disturbances, etc.

With the dawn of the high-speed computer, the same deductive process was carried a step further : different mechanisms could be combined and studied interactively, essentially non-linear processes could be studied, arbitrary initial and boundary conditions could be imposed. However, I think it would be short-sighted and narrow-minded to conclude that the utility of conventional linear methods has been eclipsed. On the contrary, numerical simulations, linear analytical methods and laboratory analogues form a very happy triumvirate for gaining insight into the physics of the atmosphere. It is perhaps in the study of the general circulation that they have shown their greatest collective versatility and compatibility. To amplify, I should first briefly reconstruct some historical perspective.

Historical remarks

The suggestion by Jeffreys (1933) that the kinematics of mid-latitude quasi-horizontal disturbances are capable of accounting for the strong poleward angular-momentum flux for the most part lay dormant until after World War II. It was at that time that Starr and his collaborators (Starr and White 1954) embarked on a programme to measure systematically these ' Jeffreys stresses ' made possible by the growing abundance of aerological data. The M.I.T. group was joined by that at U.C.L.A. (e.g., Mintz 1955) and by others. At the same time the theoretical framework for the baroclinic instability of quasi-geostrophic disturbances by Charney (1947), Eady (1949), and Fjørtoft (1951) was being developed.

The early suspicion that the quasi-horizontal cyclones and anticyclones are predominantly responsible for maintaining the heat balance at mid-latitudes was at last being confirmed. The role of the meridional circulation in the heat balance was apparently only of prime importance at low latitudes. Inadvertently the large-scale eddies also strongly transferred angular momentum poleward – this occurring mainly in the upper troposphere. For a balance to be maintained required a vertical exchange through stresses with the earth's surface which could only be accomplished by surface easterlies at low latitudes and westerlies at middle latitudes. The role of the meridional circulation was confined mainly to an effective vertical transfer of the earth's angular momentum.

The fact that the baroclinic instability process could account for the transformation of zonal potential energy to perturbation potential and kinetic energy still left an essential element lacking. How was the zonal mean kinetic energy maintained against frictional dissipation ? The answer was supplied by Onsager (1949), Fjørtoft (1953) and others, in independent studies which showed that two-dimensional and hence barotropic disturbances do *not* give a *net* transfer of energy to higher wave numbers through an inertial cascade. In fact, since the finite amplitude disturbances tend to conserve the vertical component of absolute vorticity, there is a *net* transfer to larger scales and ultimately to wave number zero – the zonal current.

The successful application of numerical techniques to simple quasi-geostrophic baroclinic models prompted Phillips (1956) to try to verify the existing body of theoretical and empirical fragments by constructing a coherent and yet simple general circulation model in a study which is now classic : a work which has been acknowledged by the Royal Meteorological Society with the award of its Napier Shaw Prize. The inevitable refinements soon followed – my own work with primitive-equation hydrodynamics (1963), and, as yet unpublished, the work of Leith permitting more refined thermal structure, the work of Mintz and Arakawa, and the new work which has been underway at our laboratory by Manabe and myself.

The first numerical models had the simple objective of trying to account for the primary processes responsible for maintaining the gross features of the atmosphere's general circulation. With the moderate success in meeting this objective, quite naturally, more detailed questions must be asked : What role does the stratosphere play and how is it coupled to the tropospheric circulation ? How sensitive are tropospheric evolutions to anomalies in the solar radiation and to the distribution of gaseous and particulate constituents of the atmosphere ? How is the hydrologic cycle coupled to the general circulation ? To what extent do the surface thermal and kinematic asymmetries determine the phase of tropospheric disturbances ? Does the lower boundary serve as a thermal flywheel so that influences from the atmosphere at one time will feed back non-linearly at some later time ? Can one estimate the determinacy with which long-range forecasts may be made ?

To answer such questions the more recent numerical models are being designed to remove many of the constraints placed on the first models and so increase the verisimilitude to the atmosphere. To enumerate some : the elimination of artificial boundaries and so deal with the global circulations – permitting one to study seasonal changes and hemispheric interactions; the reduction of truncation error by increasing horizontal resolution; the introduction of orographic variations in the lower boundary; the introduction of sufficient vertical structuring to permit the explicit calculation of radiative transfer as a function of the distribution of ozone, carbon dioxide, water vapour and cloud; the inclusion of large-scale condensation processes; the photochemical adjustments of ozone; the boundary-layer exchanges of heat, momentum, and water vapour; the internal convective transfer of heat, momentum, and water vapour; the interaction of atmosphere and ocean through their common interface. We are therefore returning to the physical complexity which Richardson (1922) felt was necessary to reproduce even short-term evolutions.

Such model generalizations carry with them increases in computational complexity of 2 or 3 orders of magnitude. Our lack of theoretical understanding of the model elements is perhaps a more serious deterrent than the lack of adequate computational apparatus. To remove a dynamical constraint or to replace a semi-empirical parametric formulation by an internally and non-linearly interactive theory, requires that the newly acquired degree of freedom account for the systematic properties of the process as well as its exceptional behaviour. Otherwise the generalization could do more harm than good. Hence we must ask ourselves whether greater detail in formulating the contributing process is warranted by truncation error, by sensitivity of the results to detail, by the increase in computational complexity and time, and by ignorance of the way these processes really work. Very often this cannot be determined in advance, but must wait for computational experiments to be performed.

Numerical methods and dissipation

Perhaps the greatest capability that numerical methods permit us is that of coping with the essential non-linearities of the atmospheric fluid system. And yet this is being recognized now as our greatest new source of difficulty in attempting to simulate the general circulation computationally. It is precisely by means of these non-linearities that the atmosphere ultimately dissipates its energy by molecular processes. Whether we work with finite difference or spectral methods, we truncate the scale beyond which we describe the physical interactions explicitly. We therefore lose the means for directly communicating with those scales where the ultimate dissipation takes place – whether it be in the boundary layer or internally in the atmosphere. It is sobering, if not discouraging, that we do not even know empirically how the energy dissipation is partitioned – much less, the processes by which the partitioning is accomplished. True, radiative-balance studies have provided us with an estimate of the total dissipation – probably to within a factor of 30 per cent. How and how much of this occurs in the surface layers, or through the intermediate processes of cumulus convection, frontal circulations, and clear-air turbulence is not known.

These sub-resolution scale processes not only involve non-linear spectral cascades of energy, but also transformations between energy components from one dimension to another, and from one form to another – that is among the latent, potential, internal, and kinetic forms. It has therefore been necessary to introduce parametrically the equivalent of small-scale lateral eddy diffusion to simulate the physically real net cascade of energy from the larger than grid-size scale to the smaller scales which have been truncated by the discrete differencing. If we assert that there is no net accumulation of energy in the sub-grid scales, then the energy removed at grid scale must be taken to be identical to the implied dissipation which must occur by molecular viscosity. In turn, the cascade in the explicit range must depend on how the differencing scheme handles the non-linear terms.

This process lies at the heart of how a numerical model will partition the dissipation of energy. Since we are able to extract very little guidance from observation as yet, the formulation of the diffusion mechanism remains a crucial unresolved problem in constructing numerical general circulation models.

Implications of certain balance properties

At the same time that one strives to construct more sophisticated general circulation models one should also look back to see what can be learned from the simpler models. The reason one calculates the details numerically by brute force is that there has been no alternative. We do not fool ourselves into believing that each five-minute time step yields details which in themselves are meaningful. The inherent indeterminacies, so eloquently revealed by the recent studies of Lorenz (1963), preclude detailed verification of numerical integrations in time and space. Most of the information content of such integrations lies in certain of their statistical properties.

Is it not reasonable to expect that the explicit calculations of the eddies which make up the Jeffreys stresses might in some way be replaced by a turbulence theory suggested by the detailed calculations ? Why not look upon numerical general circulation experiments as a laboratory apparatus with which to study and to detect simple relationships not readily deducible from the differential equations themselves or from linear causal arguments. After all, the physical and dynamical characteristics of the evolutions are readily attainable quantitatively : i.e., the mean-motion field, the eddy fluxes of heat and momentum, the energy transformations and even higher statistical moments.

What I would like to do now is to ask you to join me in taking the step backward. Let us see whether the results of a numerical experiment may suggest a simpler theoretical framework. In particular I will go to our two-level primitive equation general circulation model in which the static stability enters as a constant parameter (Smagorinsky 1963). We shall confine our attention to a time average over a number of index or energy cycles, so that the data may be regarded as describing a quasi-equilibrium state about which the ' general circulation ' is fluctuating in intensity. I justify this model as the ' laboratory ' apparatus, by noting that despite its simplicity it has succeeded in reproducing many of the observed gross characteristics of the atmosphere's general circulation.

If we map the sphere conformally onto a Mercator projection, then the earth's latitude and longitude coordinates (λ, θ) go into the map coordinates (x, y) and the map velocities are $u = \dot{x} = a\dot{\lambda}$, $v = \dot{y} = am\dot{\theta}$, in which a is the radius of the earth, $m = \sec\theta$ is the map factor, and the dot denotes individual time differentiation. It will be useful to work with zonal means

$$[\xi] \equiv \frac{1}{L}\oint \xi dx, \qquad L \equiv \oint dx = 2\pi a \qquad (1)$$

and deviations from them

$$\xi' = \xi - [\xi] . \qquad (2)$$

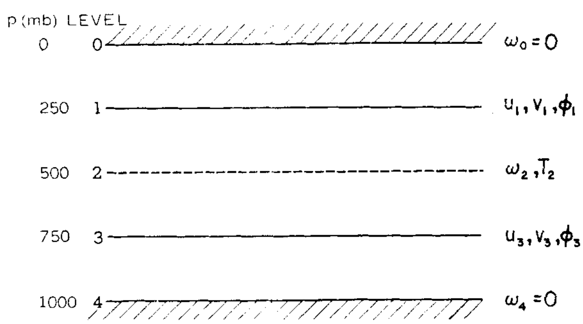

Figure 1. The two-parameter vertical structuring, showing the dependent variables at each coordinate pressure level.

With a two-parameter vertical structuring (Fig. 1) it will be convenient to deal with a measure of the first two spectral modes :

$$\left.\begin{aligned} \bar{\xi} &\equiv \xi_1 + \xi_3 \\ \hat{\xi} &\equiv \xi_1 - \xi_3 \end{aligned}\right\} . \qquad (3)$$

whence one obtains the identities for quadratic forms

$$\left.\begin{aligned} 2\,\overline{\xi v} &\equiv \bar{\xi}\,\bar{v} + \hat{\xi}\,\hat{v} \\ 2\,\widehat{\xi v} &\equiv \hat{\xi}\,\bar{v} + \bar{\xi}\,\hat{v} \end{aligned}\right\} . \qquad (4)$$

Since the temperature T_2 is defined at but one level and is related to the geopotential difference $\hat{\phi}$ by the hydrostatic relation

$$\hat{\phi} = R\,T_2 \qquad (5)$$

where R is the gas constant, then the static stability ($2\gamma^2$) enters as a parameter. We shall abbreviate the horizontal divergence and the vertical component of the relative vorticity by

$$\mathscr{D} \equiv \frac{\partial u}{\partial x} + m^2 \frac{\partial v/m^2}{\partial y} \qquad (6)$$

$$\zeta \equiv \frac{\partial v}{\partial x} - m^2 \frac{\partial u/m^2}{\partial y} \qquad (7)$$

If we filter external gravity waves by imposing the boundary conditions $\omega_0 = \omega_4 = 0$ then by virtue of the continuity equation

$$\mathscr{D} + \frac{\partial \omega}{\partial p} = 0 \qquad (8)$$

we have that

$$\overline{\mathscr{D}} \equiv 0 \qquad (9)$$

and

$$\hat{\mathscr{D}} = \frac{2\omega_2}{\hat{p}}, \quad -2\hat{p} = p_4 = 1{,}000 \text{ mb} \qquad (10)$$

We may therefore write the zonal mean of the thermodynamic energy equation, and of the two vertical modes of the zonal component of the equations of motion :

$$\frac{\partial\,[\hat{\phi}]}{\partial t} = -\gamma^2 m^2 \frac{\partial\,[\hat{v}]/m^2}{\partial y} - m^2 \frac{\partial H}{\partial y} + [\kappa Q] + [\mathscr{H}] \qquad (11)$$

$$\frac{\partial\,[\bar{u}]}{\partial t} = -\frac{m^4}{2} \frac{\partial}{\partial y} \frac{[\hat{u}]\,[\hat{v}]}{m^4} - m^4 \frac{\partial M}{\partial y} + m\,[\bar{F}_x] \qquad (12)$$

$$\frac{\partial\,[\hat{u}]}{\partial t} = \widehat{[v]\,[\zeta + f]} + m^2 \hat{V} + m\,[\hat{F}_x] \qquad (13)$$

in which we have abbreviated the quantities proportional to the non-linear eddy fluxes of heat, of angular momentum, and of relative vorticity by

$$H \equiv \frac{[\bar{v}'\,\hat{\phi}']}{2m^2} \qquad (14)$$

$$M \equiv \frac{[u'\,v']}{m^4} \qquad (15)$$

$$V \equiv \frac{[v'\,\zeta']}{m^2} \equiv -m^2 \frac{\partial M}{\partial y} + \frac{[u'\,\mathscr{D}']}{m^2} \qquad (16)$$

$f = 2\Omega \sin\theta$ is the Coriolis parameter, $[\mathscr{H}]$ and $[\mathbf{F}]/m$ represent the zonal mean eddy flux divergence of heat and momentum on a scale smaller than resolvable by the ‘ macro-scales ’ implied by $m^2\,\partial H/\partial y$ and $m^2\,\partial M/\partial y$. Q is the non-adiabatic heat rate per unit mass, and $1 - \kappa$ is the ratio of the specific heat at constant volume to that at constant pressure. The two remaining equations, for $[\bar{v}]$ and $[\hat{v}]$ have not been written down since, as we shall see, they will not be needed for our present purposes.

We shall now assume

(*i*) A quasi-equilibrium : $\partial/\partial t \equiv 0$.

(*ii*) No lateral or vertical small-scale heat diffusion : $\mathscr{H} = 0$.

(*iii*) No internal lateral or vertical small-scale momentum diffusion. The frictional forces are due only to surface boundary stresses $[\boldsymbol{\tau}_4]$:

$$[\bar{\mathbf{F}}] = -[\hat{\mathbf{F}}] = g\,[\boldsymbol{\tau}_4]/\hat{p}$$

(iv) The thermal Rossby number characterizing the flow is small compared to unity so that consequences of quasi-geostrophic balance may be imposed. This has been verified experimentally (Smagorinsky 1963) as well as observationally :

$$\text{(a)} \quad |\zeta| < f$$

$$\text{(b)} \quad \left| \frac{m^4}{2} \frac{\partial}{\partial y} \frac{[\hat{u}]\,[\hat{v}]}{m^4} \right| < \left| m^4 \frac{\partial \hat{M}}{\partial y} \right|$$

$$\text{(c)} \quad \left| \frac{[\hat{\mathscr{D}}]\,[\bar{u}]}{2} \right|, \quad \left| [\widehat{\mathscr{D}' u'}] \right| < \left| f\,[\hat{v}] \right|$$

(v) That since $|M|$ must be vanishingly small at the lower boundary and increases upward quadratically in perturbation velocity, then $|M_1| \gg |M_3|$, a property of the general circulation established by observation and reproduced by numerical experiment :

$$\hat{M} \simeq \bar{M} \simeq M_1$$

Eqs. (11), (12) and (13) therefore become :

$$\left.\begin{aligned} 0 &= -\gamma^2 m^2 \frac{\partial\,[\hat{v}]/m^2}{\partial y} - m^2 \frac{\partial H}{\partial y} + [\kappa Q] \\ 0 &= -m^4 \frac{\partial \hat{M}}{\partial y} + \frac{mg}{\hat{p}}\,[\tau_{x4}] \\ 0 &= f\,[\hat{v}] - m^4 \frac{\partial \hat{M}}{\partial y} - \frac{mg}{\hat{p}}\,[\tau_{x4}] \end{aligned}\right\} \quad . \quad . \quad (17)$$

The system of Eqs. (17) in four dependent variables $[\hat{v}]$, H, $\hat{M}$, $[\tau_{x4}]$ may be reduced to

$$\frac{\partial}{\partial y}\left(\frac{2\gamma^2}{f} m^2 \frac{\partial \hat{M}}{\partial y} + H\right) = \frac{[\kappa Q]}{m^2} \quad . \quad . \quad . \quad . \quad (18)$$

The subsidiary relations implied by the assumed momentum balance may be summarized in

$$m^2 \frac{\partial \bar{M}}{\partial y} \simeq m^2 \frac{\partial \hat{M}}{\partial y} \simeq \frac{f\,[\hat{v}]}{2m^2} \simeq -\hat{V} \simeq \frac{g\,[\tau_{x4}]}{\hat{p} m} \quad . \quad . \quad (19)$$

The result of these approximations on the maintenance of the angular momentum balance of a zonal ring is shown schematically in Fig. 2.

We are now left with the problem of producing another condition between $\hat{M}$ and H which together with Eq. (18) and appropriate lateral boundary conditions will permit us to close the argument. For this purpose we shall exploit the kinematic consequences of assuming that the baroclinic disturbances are quasi-geostrophic. If the perturbations are assumed to obey the thermal wind relation, neglecting cyclostrophic effects, i.e.

$$\left.\begin{aligned} \hat{u}' &= a\hat{\lambda}' = -\frac{m^2}{f} \frac{\partial \hat{\phi}'}{\partial y} = -\frac{m}{fa} \frac{\partial \hat{\phi}'}{\partial \theta} \\ \hat{v}' &= am\hat{\theta}' = \frac{m^2}{f} \frac{\partial \hat{\phi}'}{\partial x} = \frac{m^2}{fa} \frac{\partial \hat{\phi}'}{\partial \lambda} \end{aligned}\right\} \quad . \quad . \quad (20)$$

then upon differentiating Eq. (14) we have, with Eqs. (6) and (9)

$$\frac{1}{f} \frac{\partial H}{\partial y} \equiv \frac{1}{f} \frac{\partial}{\partial y} \frac{[\hat{\phi}'\,\bar{v}']}{2m^2} \simeq \frac{[\bar{u}'\,\hat{v}']}{2m^4} - \frac{[\hat{u}'\,\bar{v}']}{2m^4} \quad . \quad . \quad . \quad (21)$$

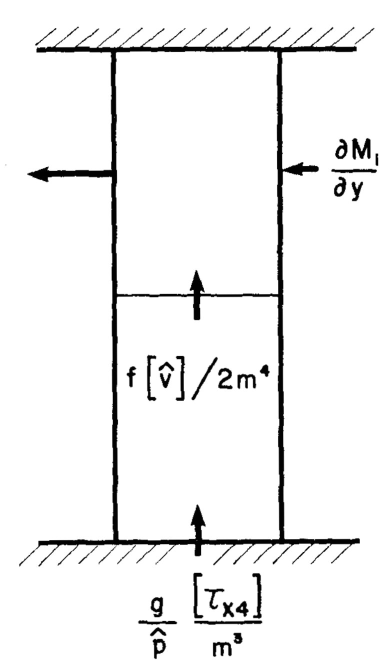

Figure 2. The processes assumed primarily to be responsible for maintaining the angular momentum balance of a zonal ring.

But Eqs. (15) and (4) yield the identity

$$\hat{M} \equiv \frac{\widehat{[u'\,v']}}{m^4} \equiv \frac{[\bar{u}'\,\hat{v}']}{2m^4} + \frac{[\hat{u}'\,\bar{v}']}{2m^4} \quad . \quad . \quad . \quad (22)$$

Hence

$$\hat{M} - \frac{1}{f}\frac{\partial H}{\partial y} = \frac{[\hat{u}'\,\bar{v}']}{m^4} \quad . \quad . \quad . \quad . \quad . \quad (23)$$

We now consider the kinematics of baroclinic disturbances which possess the minimum degrees of freedom necessary to effect a transfer of both heat and angular momentum on the sphere (analogous to the barotropic kinematics considered by Machta (1949)). Since $\bar{\mathscr{D}} \equiv 0$, then the vertically integrated flow is expressible in terms of a stream function ψ :

$$\left.\begin{aligned} \bar{u} &= a\bar{\lambda} = -\,m^2\frac{\partial\psi}{\partial y} = -\,\frac{m}{a}\frac{\partial\psi}{\partial\theta} \\ \bar{v} &= am\bar{\theta} = m^2\frac{\partial\psi}{\partial x} = \frac{m^2}{a}\frac{\partial\psi}{\partial\lambda} \end{aligned}\right\} \quad . \quad . \quad . \quad (24)$$

We permit a longitudinal phase lag, δ, between the temperature and flow perturbations, each with an angle of tilt from the meridian arctan α, and each with a zonal wave number k (Fig. 3). Hence

$$\left.\begin{aligned} \psi' &= A\sin\eta, \quad \eta = k(\lambda - \alpha\theta) \\ \hat{\phi}' &= B\sin(\eta + k\delta) \end{aligned}\right\} \quad . \quad . \quad . \quad (25)$$

More consistent perhaps would have been a spherical harmonic representation. Strictly speaking A and B must also be functions of θ so that the disturbances in general may have arbitrarily finite meridional extent. However, for our present purposes we shall assume that the amplitude variation of the perturbations is small compared with their

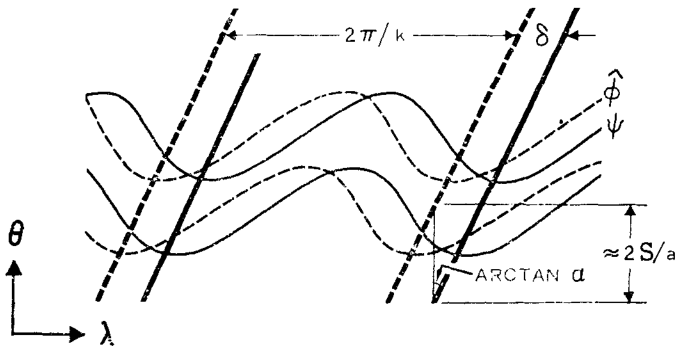

Figure 3. A schematic diagram of a simple baroclinic disturbance (stream function ψ and temperature $\hat{\phi}/R$) at 500 mb with the minimum degrees of freedom necessary to transfer both heat and momentum meridionally.

phase variation with latitude. We therefore interpret the expressions for H and $[\hat{u}'\,\bar{v}']$ which we shall calculate to apply locally, but in which A and B may be slowly varying functions of latitude. Whence we have

$$H = \frac{a}{2m}[\bar{\theta}'\,\hat{\phi}'] = \frac{ABk}{2a}[\sin(\eta + k\delta)\cos\eta] = \frac{ABk}{2a}\frac{\sin k\delta}{2}$$

and

$$\frac{[\hat{u}'\,\bar{v}']}{m^4} = \frac{a^2}{m^3}[\hat{\lambda}'\,\bar{\theta}'] = \frac{ABk^2\alpha}{a^2 mf}[\cos(\eta + k\delta)\cos\eta] = \frac{ABk^2\alpha}{a^2 mf}\frac{\cos k\delta}{2}$$

so that

$$H = f\frac{[\hat{u}'\,\bar{v}']}{m^4}mS \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

in which

$$S \equiv \frac{a \tan k\delta}{2\alpha k} \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

If the lag is small compared to the wave length ($k\delta \ll 2\pi$) and the tilt is small ($\alpha \simeq \tan\alpha \simeq \arctan\alpha$), then $2S/a \simeq \delta \operatorname{ctn}\alpha$ is independent of k and is approximately the latitudinal phase displacement (Fig. 3). Thus locally when $S \to 0$ the heat flux vanishes, and when $S \to \infty$ the angular momentum flux vanishes. This may be seen more clearly in the large by combining Eqs. (26) and (23) :

$$\hat{M} = \frac{1}{f}\left(\frac{dH}{dy} + \frac{H}{mS}\right) \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Upon integrating between two latitudinal boundaries $y = 0$, Y at which we require all fluxes to vanish, and assuming S to be a constant we have

$$S = \frac{\int_0^Y H\frac{dy}{m}}{\int_0^Y f(m\hat{M})\frac{dy}{m}} \quad . \quad . \quad . \quad . \quad . \quad (29)$$

This is even a weaker condition for the asymptotic dependence of S on H and the linear momentum flux $m\hat{M}$.

Eq. (28) represents a closing condition provided the new scale parameter S can be determined. Upon eliminating $\hat{M}$ between Eqs. (18) and (28) and integrating with the boundary conditions that H and $\hat{V}/f$ must vanish, we have

$$\frac{2\gamma^2 m^2}{f}\frac{d}{dy}\left\{\frac{1}{f}\left(\frac{dH}{dy}+\frac{H}{mS}\right)\right\}+H=q \quad . \quad . \quad (30)$$

where

$$q(y)=\int_0^y \frac{[\kappa Q]}{m^2}\,dy \quad . \quad . \quad . \quad . \quad . \quad (31)$$

is the total heat flux demanded by the external heating at a given latitude. For equilibrium we see from Eq. (17) that $q(0)=q(y)=0$ in order to satisfy the boundary conditions on $[\hat{v}]$ and H.

At this point we note that the linear-nonhomogeneous second-order ordinary differential Eq. (30) together with the boundary conditions

$$H=0 \quad \text{on} \quad y=0, Y \quad . \quad . \quad . \quad . \quad . \quad (32)$$

form a mathematically determined system for an arbitrary specification of the parameters $2\gamma^2$ and S. However, we have yet to impose the physical boundary conditions on the angular momentum flux :

$$\hat{M}=0 \quad \text{on} \quad y=0, Y \quad . \quad . \quad . \quad . \quad . \quad (33)$$

which by virtue of Eqs. (28) and (32) require

$$\frac{dH}{dy}=0 \quad \text{on} \quad y=0, Y \quad . \quad . \quad . \quad . \quad . \quad (34)$$

These two new subsidiary conditions now fix the values of $2\gamma^2$ and S for which equilibrium solutions exist. For the sake of completeness we shall record certain integral consequences of Eq. (33). Upon integrating Eq. (18) twice we have

$$\int_0^Y \frac{f}{m^2}(q-H)\,dy=0 \quad . \quad . \quad . \quad . \quad . \quad (35)$$

Furthermore upon applying Eqs. (33) to (19) we obtain

$$\int_0^Y \frac{f\,[\hat{v}]}{m^4}\,dy=\int_0^Y \hat{V}\,\frac{dy}{m^2}=\int_0^Y \frac{[\tau_{x4}]}{m^3}\,dy=0 \quad . \quad . \quad (36)$$

The system (30), (32) and (34) will be solved numerically for equal increments in $y=j\Delta$, $0\leqslant j\leqslant(J-1)$ with $\Delta=555$ km corresponding to 5 degrees of longitude at the equator. Ordinary central space differences are employed. The difference equation becomes

$$-C_j H_{j+1}+D_j H_j-E_j H_{j-1}=q_j$$

in which

$$\left.\begin{aligned} C_j &= -\frac{2\gamma^2}{\Delta}\left(\frac{m^2}{f}\right)_j\frac{1}{f_{j+\frac{1}{2}}}\left(\frac{1}{\Delta}+\frac{1}{2Sm_{j+1}}\right)\\ D_j &= \frac{2\gamma^2}{\Delta}\left(\frac{m^2}{f}\right)_j\left(\frac{-\frac{1}{\Delta}+\frac{1}{2Sm_j}}{f_{j+\frac{1}{2}}}-\frac{\frac{1}{\Delta}+\frac{1}{2Sm_j}}{f_{j-\frac{1}{2}}}\right)+1\\ E_j &= -\frac{2\gamma^2}{\Delta}\left(\frac{m^2}{f}\right)_j\frac{1}{f_{j-\frac{1}{2}}}\left(\frac{1}{\Delta}-\frac{1}{2Sm_{j-1}}\right)\end{aligned}\right\} \quad . \quad . \quad (37)$$

with terminal conditions

$$H_0=H_{J-1}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

and side conditions

$$H_1=H_{J-2}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

The recursion Eq. (37) with (38) may be solved exactly by the method given by Richtmyer (1957). The applicability of this method is subject to restrictions on the coefficients C_j, D_j, E_j, which apparently were satisfied in the case to be discussed here. For a particular $q(y)$ one may find solutions of Eqs. (37) and (38) for values of $2\gamma^2$ and S which also satisfy Eq. (39). This may be done crudely by a systematic trial and error procedure and then refined by numerical interpolation.

An attempt has been made (Smagorinsky 1963) to parameterize the non-adiabatic heating for a vertically integrated atmosphere from recent calculations by London (1957) and Budyko (1956). It was shown that after accounting for the latent heat transfer, for the condensation-evaporation difference, and for the transport by the oceans, the required sensible-heat transport by atmospheric adiabatic dynamics differs considerably from the purely radiative requirements computed by Houghton (1954) and London (1957). The comparison is reproduced in Fig. 4.

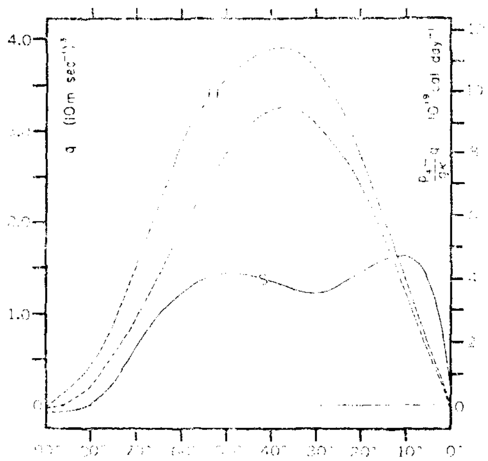

Figure 4. The annual mean heat transport required to balance the radiation excess calculated by Houghton (1954) (H) and London (1957) (L); the required annual mean sensible heat transport by atmospheric adiabatic dynamics calculated by Smagorinsky (1963) (S). The latter was used in the present calculations.

If we take the poleward boundary at 80·9° latitude, approximately where q vanishes on curve S in Fig. 4, then $J = 30$. The numerical integration yields the characteristic values $2S/a = 0{\cdot}914$ and $2\gamma^2/2\gamma_s^2 = 0{\cdot}967$, where $2\gamma_s^2 = 8250\ \text{m}^2\,\text{sec}^{-2}$ corresponds to the static stability of the standard atmosphere (equivalent to a lapse rate of 6·5 deg km^{-1}). These characteristic values and the corresponding solutions were found to be unique within the range $0 \leqslant 2S/a \leqslant 300$ and $0{\cdot}33 \leqslant 2\gamma^2/2\gamma_s^2 \leqslant 4{\cdot}0$. We have assumed that the existence of possible solutions for smaller static stabilities would be outside of the range of validity of the geostrophic regime.

A few words should be said on the implications of the occurrence of a unique static stability. The general range of tropospheric static stability has been found to be deducible from purely vertical transfer processes – radiation and convection (Manabe and Möller, 1961). However, within this range we found that the quasi-geostrophic dynamical processes and the net heating fix the value for which an equilibrium state can exist. It would thus appear that a time-dependent model which requires a prescription of the static stability (e.g., Smagorinsky 1963) cannot in general be expected to asymptotically approach a non-trivial equilibrium.

Although the mean temperature profile (and therefore the zonal thermal wind) have nowhere entered explicitly into the preceding development, the characteristic static stability permits an estimate of the meridional mean temperature gradient. It has been hypothesized (Smagorinsky 1963) from quasi-geostrophic baroclinic instability theory that the general circulation will fluctuate about an equilibrium characterized by a state in which the meridional slope of the isentropes is a minimum. Hence the domain mean zonal thermal wind is proportional to the static stability, in particular,

$$\hat{U} \sim \frac{2\gamma^2}{af}$$

Taking $f \sim 10^{-4}$ sec^{-1}, the characteristic value of $2\gamma^2$ yields $\hat{U} \sim 13$ m sec^{-1}. Furthermore the two-dimensional wave number corresponding to such an equilibrium about a minimum slope of isentropes was found to be

$$n \sim 2^{1/4} \frac{af}{\sqrt{2\gamma^2}}$$

which yields $n \sim 8{\cdot}5$. Hence S and n together determine some of the scale properties of the wave disturbances which are maintaining the heat and angular momentum balance.

The solution for the eddy heat flux is given in Fig. 5. From this one may calculate the poleward fluxes of heat by the mean meridional circulation, of angular momentum and relative vorticity by the eddies, and the surface stress from

$$\frac{\gamma^2 [\hat{v}]}{m^2} = (q - H) \quad . \quad . \quad . \quad . \quad . \quad (40)$$

$$\hat{M} = \frac{1}{2\gamma^2} \int_0^y \frac{f}{m^2} (q - H)\, dy \quad . \quad . \quad . \quad . \quad (41)$$

$$\hat{V} = -\frac{f}{2\gamma^2}(q - H) \quad . \quad . \quad . \quad . \quad (42)$$

$$[\tau_{x4}] = \frac{\hat{p}\, mf}{g 2\gamma^2}(q - H) \quad . \quad . \quad . \quad . \quad . \quad (43)$$

which are shown in Figs. 6, 7, 8, and 9, respectively.

From Fig. 6 we see that equilibrium requires a three-cell mean meridional circulation with internal nodes at 32° and 62° latitude corresponding, respectively, to maximum subsidence in the surface sub-tropical anticyclone and to the systematic ascending motion in the polar-front zone. For a more direct verification against observation we note that H (Fig. 5) agrees quite well with sensible heat flux measurements taken from observation. The main discrepancy lies in a systematic equatorward shift in the computed profile, it being worst at high latitudes. Similarly $\hat{M}$ and $[\tau_{x4}]$ verify favourably, the most serious discrepancy occurring poleward of 50° latitude. In part this is attributable to the uncertainty in establishing the heating function in Fig. 4. This together with the influence of the boundary at 81° latitude evidently distorts the role of poleward direct circulation, it probably being too intense and too far equatorward. The result is a small equatorward angular momentum flux at high latitudes as well as negative surface stress which would require surface easterlies poleward of 62° latitude.

For an equilibrium state it appears that for the order of approximation imposed here, the static stability, the eddy fluxes of heat, momentum, and shear vorticity, the mean meridional circulation and the surface stress distribution may be determined from the radiative heating gradient without any explicit reference to the temperature or zonal wind field or to a prescription as to the functional dependence of the surface stress on the surface wind. In fact, the temperature and zonal wind remain undetermined unless we appeal to the theory of unstable baroclinic waves. This therefore differs in essence from classical turbulence theories constructed about *austausch* frameworks. Then again this is not too surprising, since attempts to apply classical techniques to the general circulation have not been successful.

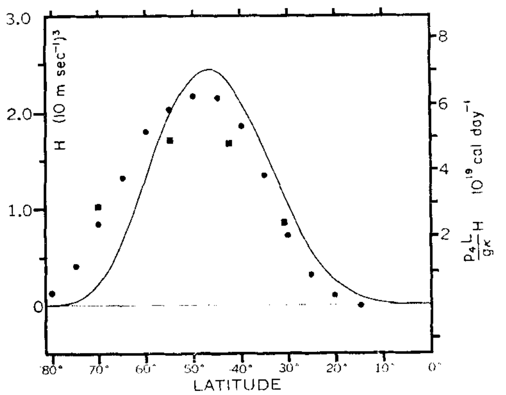

Figure 5. The calculated eddy heat flux. Annual mean observed values : ■ Starr and White (1954); ● Mintz (1955).

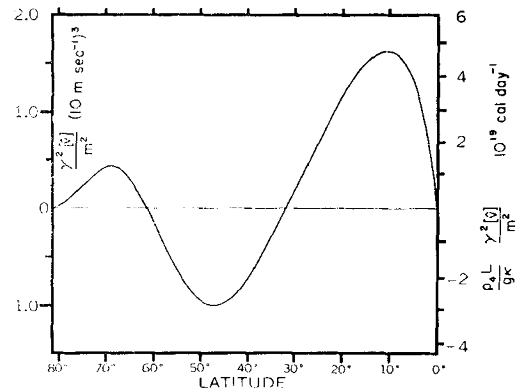

Figure 6. The calculated heat flux by the mean meridional circulation.

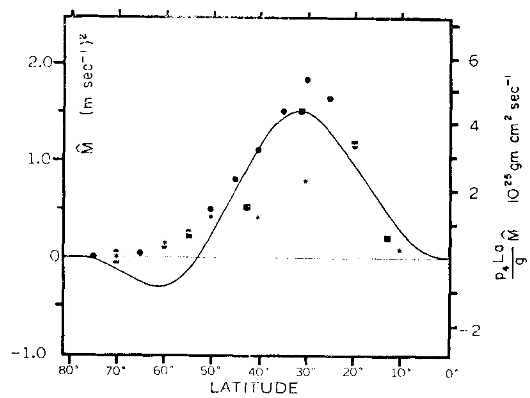

Figure 7. The calculated shear eddy angular momentum flux. Annual mean observed values : ■ Starr and White (1954); ★ Buch (1954); ● Mintz (1955).

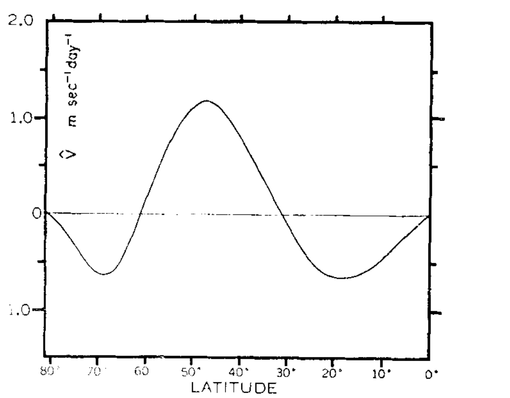

Figure 8. The calculated shear eddy relative vorticity flux.

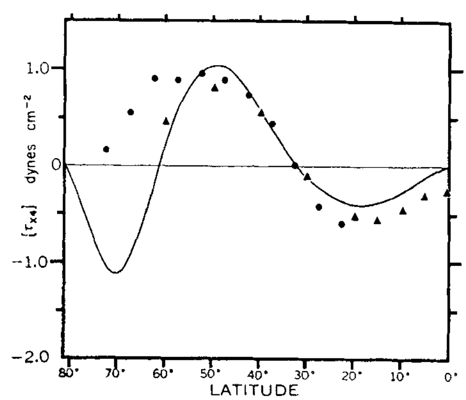

Figure 9. The calculated surface stress. Annual mean observed values : ▲ Priestley (1951); ● Mintz (1955).

One would hope this suggests that an appropriate turbulence theory could ultimately be developed for the general circulation. Some of the obvious questions raised are : Why are the temperature and zonal wind profiles irrelevant to this order of approximation in determining the eddy flux characteristics ? Is it possible to develop a higher order theory ? Here we have been concerned with the equilibrium state – is it possible to extend these notions to a time-dependent theory ?

Concluding remarks

Returning now to the general theme of this lecture, I would like to say that the future looks bright and promising – but not without its pitfalls. The general circulation by its very nature depends upon integrating those fields of knowledge which in the view of some have stood on the fringes of meteorology. At present, the large gaps in our understanding of the contributing physical processes prevent the construction of fully coherent general circulation models. However, I hope that this difficulty will serve as a cohesive influence in uniting the various disciplines to at least one obvious common purpose. It will at the same time require individual workers to broaden their outlooks thereby reversing the trend toward specialization.

We must guard against equating the massive outputs of high-speed computers with understanding. The computer at best is a very convenient laboratory tool – but it is not the end in itself. The design of experiments and the devising of perceptive methods for diagnosing and interpreting the results are still quite primitive. However, experience in the past few years indicates that numerical methods potentially have an elegance comparable to that of traditional analytical methods – but that its full realization is yet to be achieved.

Finally, as we isolate the essential processes responsible for the characteristics of the general circulation, ultimately one would expect to be able to dispense with the unnecessary and irrelevant detail – thereby reversing the trend toward more complex models and larger computers.

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